~~W4 tonight 23:59~~
~~HW4 Wed Apr 1~~

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6.2 Transforms of derivatives, and ODEs

6.2.1 Transforms of derivatives

In a sense, the Laplace transform method lets us replace "calculus" with "algebra"

$$\boxed{\mathcal{L}\{g'(t)\} = s\,G(s) - g(0)} \quad \textbf{where } G(s) = \mathcal{L}\{g(t)\}$$

We replace "take the derivative" with "multiply by s" plus an adjustment depending on initial value.

Proof:

$$\mathcal{L}\{g'(t)\} = \int_0^\infty e^{-st} g'(t)\,dt \quad (\text{definition})$$

$$= e^{-st} g(t)\Big|_0^\infty + \int_0^\infty s e^{-st} g(t)\,dt \quad (\text{integrate by parts})$$

$$= \underbrace{\lim_{t\to\infty} e^{-st} g(t)}_{0} - g(0) + s\underbrace{\int_0^\infty e^{-st} g(t)\,dt}_{G(s) = \mathcal{L}\{g(t)\}}$$

$$= -g(0) + s\,G(s)$$

provided $g(t)$ is of exponential order so that $\lim_{t\to\infty} e^{-st} g(t) = 0$.

(Optional detail: if $g(t)$ is of exponential order, there exist M, c, t_0 such that

$$|g(t)| \le Me^{ct} \text{ for all } t \ge \quad \Leftrightarrow \quad -Me^{ct} \le g(t) \le Me^{ct} \text{ for all } t \ge t_0$$

$$\Rightarrow -M\underbrace{\lim_{t\to\infty} e^{-(s-c)t}}_{=0 \text{ if } s>c} = \lim_{t\to\infty} e^{-st}(-Me^{ct}) \le \lim_{t\to\infty} e^{-st} g(t) \le \lim_{t\to\infty} e^{-st}(Me^{ct}) = M\underbrace{\lim_{t\to\infty} e^{-(s-c)t}}_{=0 \text{ if } s>c}$$

so $\lim_{t\to\infty} e^{-st} g(t) = 0$ if $s > c$ by the "Squeeze Theorem" [CLP1 Thm. 1.4.17])

<u>Exercise</u> Prove the useful formula

$$\boxed{\mathcal{L}\{g''(t)\} = s^2 G(s) - s\,g(0) - g'(0)}$$

where $G(s) = \mathcal{L}\{g(t)\}$.

Following this pattern, we could prove

$$\mathcal{L}\{g'''(t)\} = s^3 G(s) - s^2 g(0) - s\,g'(0) - g''(0)$$

etc., i.e.

$$\mathcal{L}\{g^{(n)}(t)\} = s^n G(s) - s^{n-1} g(0) - \dots - g^{(n-1)}(0)$$

for $n = 1, 2, 3, \dots$

See Table 6.2, p. 300.

Using transforms of derivatives, we can transform a <u>differential</u> equation for $x(t)$ into an <u>algebraic</u> equation for its transform $X(s) = \mathcal{L}\{x(t)\}$.

Then we use <u>algebra</u> to solve explicitly for $X(s)$, and then take the <u>inverse transform</u> to get the solution of the differential equation

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$

6.2.2 Solving ODEs with the Laplace transform

Example 6.2.A Use Laplace transforms to solve the IVP

$$x'' + x = \cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$$

Let $X(s) = \mathcal{L}\{x(t)\} \left(= \int_0^\infty e^{-st} x(t)\, dt \right)$

Take the Laplace transform of the entire IVP, use linearity (and Tables 6.1, 6.2)

$$\underbrace{\mathcal{L}\{x''(t)\} + \mathcal{L}\{x(t)\}} = \mathcal{L}\{\cos(2t)\}$$

$$s^2 X(s) - s\underbrace{x(0)}_{0} - \underbrace{x'(0)}_{1} + X(s) = \frac{s}{s^2+2^2}$$

Solve for $X(s)$

$$s^2 X(s) - 1 + X(s) = \frac{s}{s^2+4}$$

$$(s^2+1)\, X(s) = \frac{s}{s^2+4} + 1$$

$$X(s) = \frac{s}{(s^2+1)(s^2+4)} + \underbrace{\frac{1}{s^2+1}}_{\text{in Table 6.1}}$$

(s^2+1, s^2+4 are both irreducible quadratics (do not factor over the reals) and are already in completed-square form)

Before we can find $x(t) = \mathcal{L}^{-1}\{X(s)\}$, we need some algebra

Partial fractions:

$$\frac{s}{(s^2+1)(s^2+4)} = \frac{As+C}{s^2+1} + \frac{Bs+D}{s^2+4}$$

(Can be done more simply if you have insight, but this is general method - see CLP2)

$$s = (As+C)(s^2+4) + (Bs+D)(s^2+1)$$

$$0s^3 + 0s^2 + s + 0 = \underbrace{(A+B)}_{=0}s^3 + \underbrace{(C+D)}_{=0}s^2 + \underbrace{(4A+B)}_{=1}s + \underbrace{(4C+D)}_{=0}$$

$$\left.\begin{aligned} C+D &= 0 \\ 4C+D &= 0 \end{aligned}\right\} \Leftrightarrow C=0, D=0 \qquad \left.\begin{aligned} A+B &= 0 \\ 4A+B &= 1 \end{aligned}\right\} \Leftrightarrow A=\frac{1}{3}, B=-\frac{1}{3}$$

$$\frac{s}{(s^2+1)(s^2+4)} = \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4}$$

Using this, we get

$$X(s) = \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4} + \frac{1}{s^2+1}$$

Now we can use Table 6.1 to find inverse transform

$$x(t) = \frac{1}{3}\cos(t) - \frac{1}{3}\cos(2t) + \sin(t)$$

<u>Exercise</u> Solve the same IVP using

(a) the method of undetermined coefficients

(b) the variation of parameters method

Seeing this example, you might think the Laplace transform method for solving IVPs is less efficient than other methods for 2nd order linear ODEs. The Laplace transform method has advantages when the nonhomogeneous (or "forcing") term is more general, e.g. discontinuous.