6.2.3 Using the Heaviside function

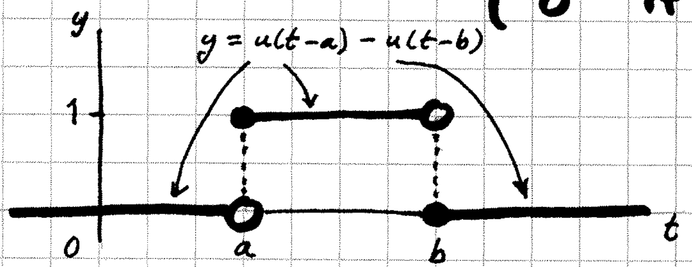
We use $u(t-a)$ to re-express piecewise defined functions.

(Recall $u(t-a) = \begin{cases} 0 & \text{if } t<a \\ 1 & \text{if } t \geq a \end{cases}$, $\mathcal{L}\{u(t-a)\} = \dfrac{e^{-as}}{s}$)

Example 6.2.B "switch on at $t=a$, switch off at $t=b$"

Let $0 \leq a < b$. Then:

$$u(t-a) - u(t-b) = \begin{cases} 0 & \text{if } t<a & (0-0, \text{ since } t<a \ \& \ t<b) \\ 1 & \text{if } a \leq t < b & (1-0, \text{ since } t \geq a \ \& \ t<b) \\ 0 & \text{if } t \geq b & (1-1, \text{ since } t \geq a \ \& \ t \geq b) \end{cases}$$

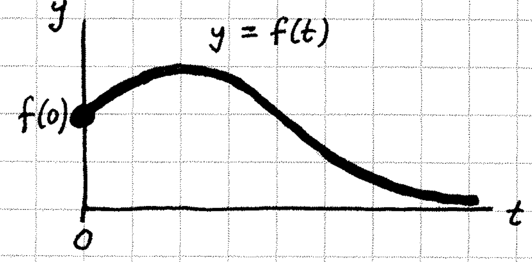


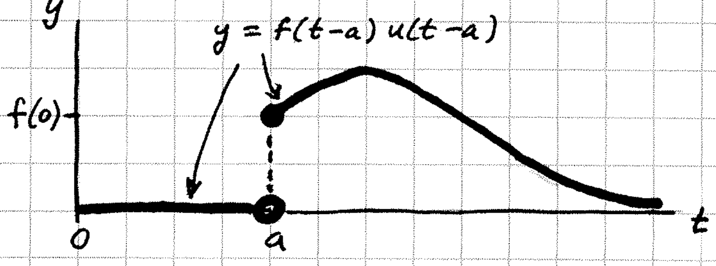
$$\mathcal{L}\{u(t-a) - u(t-b)\} = \frac{e^{-as} - e^{-bs}}{s}$$

Translation in t: if $f(t)$, $t \geq 0$, then its translation to the right by a units is

$$f(t-a)\,u(t-a) = \begin{cases} 0 & \text{if } t<a & (f(t-a) \cdot 0, \text{ since } t<a) \\ f(t-a) & \text{if } t \geq a & (f(t-a) \cdot 1, \text{ since } t \geq a) \end{cases}$$

e.g.





Theorem (Second shifting property ; "t-shift")

$$\boxed{\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}F(s)}$$

where $F(s) = \mathcal{L}\{f(t)\}$.

Proof:

$$\begin{aligned}
\mathcal{L}\{f(t-a)\,u(t-a)\} &= \int_0^\infty e^{-st} f(t-a)\,u(t-a)\,dt \\
&= \int_0^a e^{-st} f(t-a)\underbrace{u(t-a)}_{0}\,dt + \int_a^\infty e^{-st} f(t-a)\underbrace{u(t-a)}_{1}\,dt \\
&= \int_a^\infty e^{-st} f(t-a)\,dt \quad (\tau = t-a,\ t = \tau + a,\ dt = d\tau) \\
&= \int_0^\infty e^{-s(\tau+a)} f(\tau)\,d\tau \\
&= e^{-sa}\int_0^\infty e^{-s\tau} f(\tau)\,d\tau \quad (\text{dummy variable } \tau) \\
&= e^{-as}\underbrace{\int_0^\infty e^{-st} f(t)\,dt}_{=\mathcal{L}\{f(t)\} = F(s)}
\end{aligned}$$