HW5 Wed Apr 1

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Example 6.2.C Solve the IVP, write the solution as a simplified piecewise defined function, draw the graph of the solution for $t \geq 0$:

$$x'' + x = f(t), \quad x(0)=1, \quad x'(0)=0, \quad \text{where } f(t) = \begin{cases} 0 & \text{if } t < \pi \\ 4 & \text{if } \pi \leq t < 2\pi \\ 0 & \text{if } t \geq 2\pi \end{cases}$$

(i.e. $f(t) = 4$ if $\pi \leq t < 2\pi$, $f(t) = 0$ otherwise)

1. Write $f(t)$ in terms of Heaviside function $u(t)$
2. Laplace transform the IVP, solve for $X(s) = \mathcal{L}\{x(t)\}$
3. Find inverse transform $x(t) = \mathcal{L}^{-1}\{X(s)\}$

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(1) See Example 6.2.B. "rectangular pulse"

$$f(t) = 4\left[u(t-\pi) - u(t-2\pi)\right] = 4u(t-\pi) - 4u(t-2\pi)$$

(2)

$$s^2 X(s) - s\underbrace{x(0)}_{=1} - \underbrace{x'(0)}_{=0} + X(s) = 4\frac{e^{-\pi s}}{s} - 4\frac{e^{-2\pi s}}{s}$$

$$(s^2+1)X(s) - s = 4\frac{e^{-\pi s}}{s} - 4\frac{e^{-2\pi s}}{s}$$

$$X(s) = 4e^{-\pi s}\frac{1}{s(s^2+1)} - 4e^{-2\pi s}\frac{1}{s(s^2+1)} + \underbrace{\frac{s}{s^2+1}}_{\text{in Table 6.1}}$$

(3)

$$x(t) = \mathcal{L}^{-1}\left\{4e^{-\pi s}\frac{1}{s(s^2+1)} - 4e^{-2\pi s}\frac{1}{s(s^2+1)} + \frac{s}{s^2+1}\right\}$$

- Table 6.1: $\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos(t)$
- $e^{-\pi s}$, $e^{-2\pi s}$: 2nd shifting property applied to $\frac{1}{s(s^2+1)}$
- $\frac{1}{s(s^2+1)}$: partial fractions, then Table 6.1

Partial fractions:

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$1 = A(s^2+1) + (Bs+C)s$$

$$= \underbrace{(A+B)}_{=0} s^2 + \underbrace{C}_{=0} s + \underbrace{A}_{=1} \iff \begin{cases} A = 1 \\ B = -1 \\ C = 0 \end{cases}$$

so

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1} \quad ; \quad \mathcal{L}\{1\} = \frac{1}{s}, \ \mathcal{L}\{\cos(t)\} = \frac{s}{s^2+1}$$

therefore

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos(t)$$

$$\mathcal{L}^{-1}\left\{4e^{-\pi s}\frac{1}{s(s^2+1)}\right\} = 4[1-\cos(t-\pi)]u(t-\pi)$$

$$\mathcal{L}^{-1}\left\{4e^{-2\pi s}\frac{1}{s(s^2+1)}\right\} = 4[1-\cos(t-2\pi)]u(t-2\pi)$$

and

$$x(t) = 4[1-\cos(t-\pi)]u(t-\pi) - 4[1-\cos(t-2\pi)]u(t-2\pi) + \cos(t)$$

Write as piecewise defined:

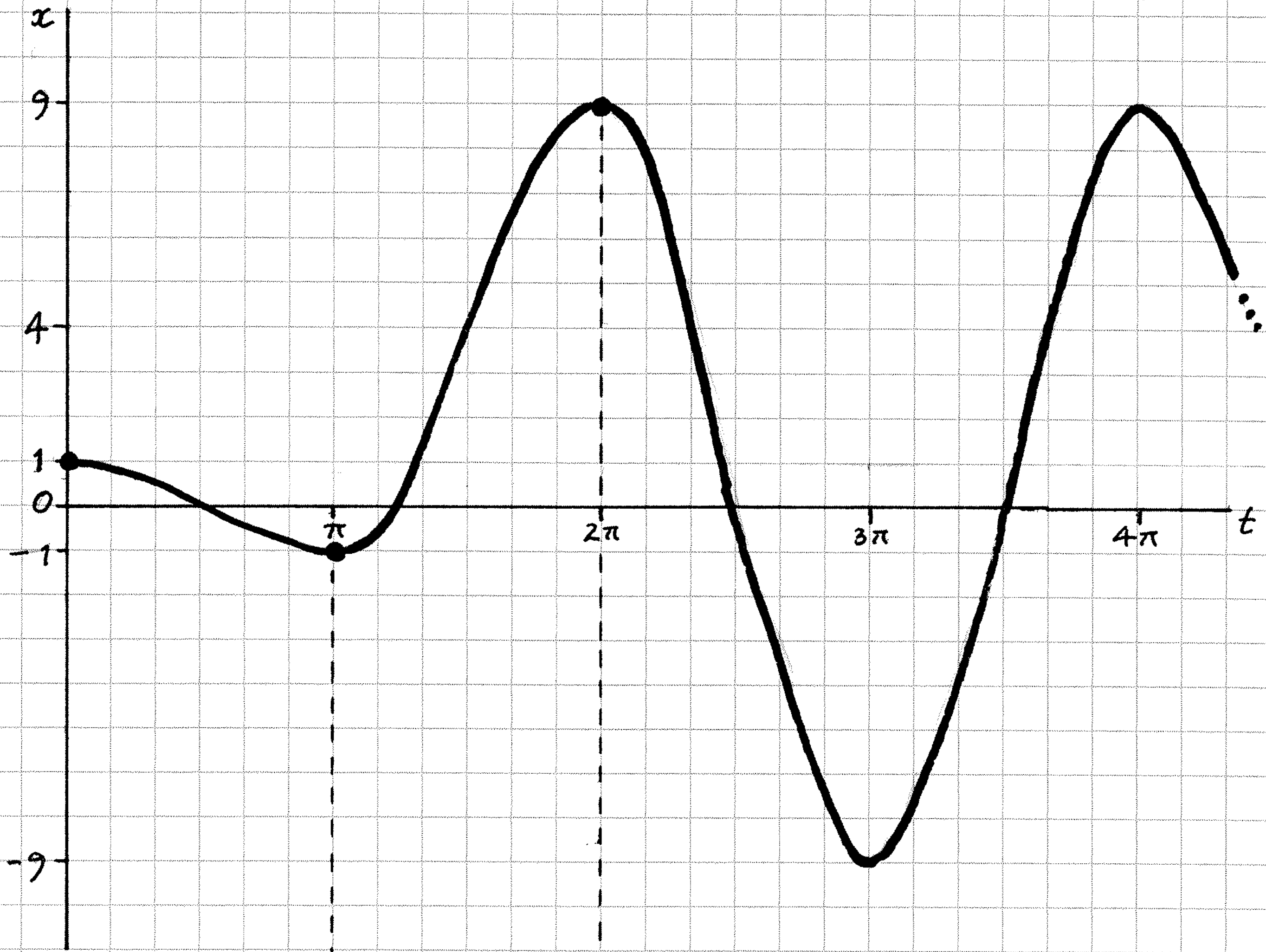
(because if $t<\pi$ then $u(t-\pi)=0$ and $u(t-2\pi)=0$ etc.)

$$x(t) = \begin{cases} \cos(t) & \text{if } t<\pi \\ \cos(t) + 4[1-\overbrace{\cos(t-\pi)}^{=-\cos(t)}] & \text{if } \pi \le t < 2\pi \\ \cos(t) + 4[1-\underbrace{\cos(t-\pi)}_{=-\cos(t)}] - 4[1-\underbrace{\cos(t-2\pi)}_{=\cos(t)}] & \text{if } t \ge 2\pi \end{cases}$$

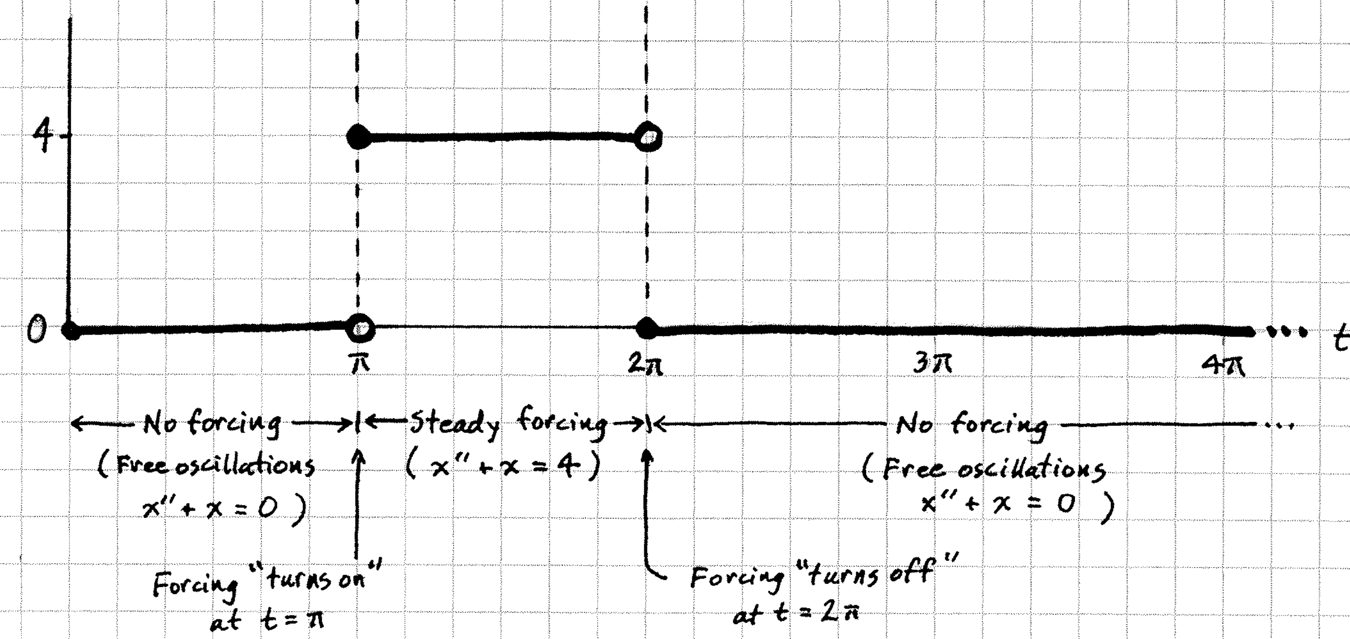
and simplify:

$$x(t) = \begin{cases} \cos(t) & \text{if } t<\pi \\ 4 + 5\cos(t) & \text{if } \pi \le t < 2\pi \\ 9\cos(t) & \text{if } t \ge 2\pi \end{cases}$$

Graph of t vs. $x(t)$ for $t \geq 0$: (notice $x(t)$ is continuous)



It is instructive to compare with the graph of the rectangular pulse forcing $f(t)$ and think of an undamped mass and spring system.



Details:

$$x(t) = \cos(t) + 4[1-\cos(t-\pi)]\,u(t-\pi) - 4[1-\cos(t-2\pi)]\,u(t-2\pi)$$

Because of $u(t-\pi)$, $u(t-2\pi)$ we change expressions at $t=\pi$ and $t=2\pi$, so write as

$$x(t) = \begin{cases} \cos(t) + 4[1-\cos(t-\pi)]u(t-\pi) - 4[1-\cos(t-2\pi)]u(t-2\pi) & \text{if } t<\pi \\ \cos(t) + 4[1-\cos(t-\pi)]u(t-\pi) - 4[1-\cos(t-2\pi)]u(t-2\pi) & \text{if } \pi \le t < 2\pi \\ \cos(t) + 4[1-\cos(t-2\pi)]u(t-\pi) - 4[1-\cos(t-2\pi)]u(t-2\pi) & \text{if } t \ge 2\pi \end{cases}$$

Then sub. in correct values for $u(t-\pi)$, $u(t-2\pi)$ on each interval:

$$x(t) = \begin{cases} \cos(t) + 4[1-\cos(t-\pi)]\,0 - 4[1-\cos(t-2\pi)]\,0 & \text{if } t<\pi \\ \cos(t) + 4[1-\cos(t-\pi)]\,1 - 4[1-\cos(t-2\pi)]\,0 & \text{if } \pi \le t < 2\pi \\ \cos(t) + 4[1-\cos(t-\pi)]\,1 - 4[1-\cos(t-2\pi)]\,1 & \text{if } t \ge 2\pi \end{cases}$$

and simplify

$$x(t) = \begin{cases} \cos(t) & \text{if } t<\pi \\ \cos(t) + 4[1-\cos(t-\pi)] & \text{if } \pi \le t < 2\pi \\ \cos(t) + 4[1-\cos(t-\pi)] - 4[1-\cos(t-2\pi)] & \text{if } t \ge 2\pi \end{cases}$$

Then use trig identities $\cos(t-\pi) = -\cos(t)$, $\cos(t-2\pi) = \cos(t)$ [could do earlier]

$$x(t) = \begin{cases} \cos(t) & \text{if } t<\pi \\ \cos(t) + 4[1+\cos(t)] & \text{if } \pi \le t < 2\pi \\ \cos(t) + 4[1+\cos(t)] - 4[1-\cos(t)] & \text{if } t \ge 2\pi \end{cases}$$

Finally, expand each line and simplify again

$$x(t) = \begin{cases} \cos(t) & \text{if } t<\pi \\ 4 + 5\cos(t) & \text{if } \pi \le t < 2\pi \\ 9\cos(t) & \text{if } t \ge 2\pi \end{cases}$$

When you get enough practice (e.g. fill in missing steps in lectures, etc.) you won't have to write so many steps explicitly and save some time.