Example 6.2.D Find $\mathcal{L}^{-1}\left\{\frac{1-e^{-2s}}{s^2}\right\}$ written as a simplified piecewise defined function, and draw its graph for $t \geq 0$.

$$\frac{1-e^{-2s}}{s^2} = \frac{1}{s^2} - e^{-2s}\frac{1}{s^2}$$

Table 6.1:

$$\mathcal{L}\{t\} = \frac{1}{s^2} \quad \text{so} \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

- e^{-2s} : 2nd shifting property

Let $f(t) = t$, $F(s) = \frac{1}{s^2}$

By the 2nd shifting property $\mathcal{L}\{f(t-2)u(t-2)\} = e^{-2s}F(s)$

$$\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s^2}\right\} = f(t-2)u(t-2) = (t-2)u(t-2)$$

Therefore

$$\mathcal{L}^{-1}\left\{\frac{1-e^{-2s}}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s^2}\right\}$$

$$= t - (t-2)u(t-2)$$

Written as piecewise defined

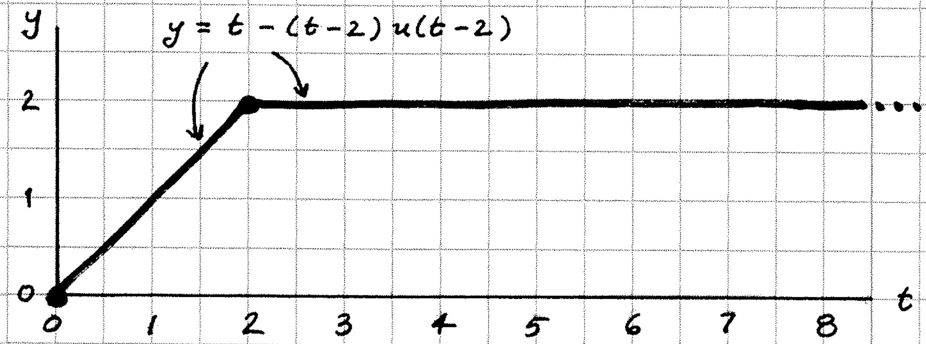
$$t - (t-2)\underbrace{u(t-2)}_{=0 \text{ or } 1} = \begin{cases} t - (t-2)\cdot 0 & \text{if } t < 2 \\ t - (t-2)\cdot 1 & \text{if } t \geq 2 \end{cases}$$

simplified

$$\mathcal{L}^{-1}\left\{\frac{1-e^{-2s}}{s^2}\right\} = \begin{cases} t & \text{if } t < 2 \\ 2 & \text{if } t \geq 2 \end{cases}$$

This type of function is called a "ramp"

Graph



Example 6.2.E Solve the IVP, write the solution as a simplified piecewise defined function, draw the graph of the solution for $t \geq 0$:

$$x'' + 4x = f(t),\quad x(0)=0,\ x'(0)=0, \text{ where } f(t) = \begin{cases} t & \text{if } 0 \leq t < \pi \\ \pi & \text{if } t \geq \pi \end{cases}$$

"ramp forcing"

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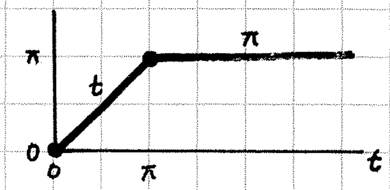
1 From Example 6.2.D, $f(t)$ written in terms of the Heaviside function is

$$f(t) = t - (t-\pi)\,u(t-\pi)$$

Exercise. Verify.

(Graphically :



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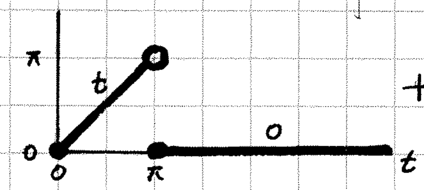

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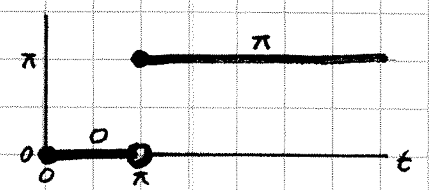

$$f(t) = t\,\underbrace{[u(t-0) - u(t-\pi)]}_{\substack{\text{turn on at } t=0,\ \text{turn off at } t=\pi \\ \text{(Example 6.2.B)}}} + \pi\,u(t-\pi)$$

$$= t\,\underbrace{u(t-0)}_{=1 \text{ for } t \geq 0} - (t-\pi)\,u(t-\pi)$$

we only care about $t \geq 0$)