~~HW5 due Wed Apr 1~~ 33 Mon 2020-03-30

6.2.3 continued

Example 6.2.E continued

1 $x''+4x = \underline{t-(t-\pi)u(t-\pi)}$, $x(0)=0$, $x'(0)=0$

2 **Next, take Laplace transforms, and algebraically solve for the transform of the solution.**

Let $\mathcal{L}\{x(t)\} = X(s)$

$$\mathcal{L}\{x''(t)\} + 4\mathcal{L}\{x(t)\} = \mathcal{L}\{t\} - \underbrace{\mathcal{L}\{(t-\pi)u(t-\pi)\}}_{\text{see Ex. 6.2.D}}$$

$$s^2X(s) - s\underbrace{x(0)}_{=0} - \underbrace{x'(0)}_{=0} + 4X(s) = \frac{1}{s^2} - e^{-\pi s}\frac{1}{s^2}$$

$$s^2X(s) + 4X(s) = \frac{1}{s^2} - e^{-\pi s}\frac{1}{s^2}$$

$$X(s) = \frac{1}{s^2(s^2+4)} - e^{-\pi s}\frac{1}{s^2(s^2+4)}$$

3 **Find the inverse transform** $x(t) = \mathcal{L}^{-1}\{X(s)\}$

- partial fractions: $\dfrac{1}{s^2(s^2+4)}$
- second shifting property: $e^{-\pi s}F(s)$

Partial fractions

$$\frac{1}{s^2(s^2+4)} = \underbrace{\frac{A}{s} + \frac{B}{s^2}}_{\text{from general linear factor}} + \underbrace{\frac{Cs+D}{s^2+4}}_{\text{from simple (irreducible) quadratic factor}}$$

$$1 = As(s^2+4) + B(s^2+4) + (Cs+D)s^2$$

$$1 = \underbrace{(A+C)}_{=0}s^3 + \underbrace{(B+D)}_{=0}s^2 + \underbrace{4A}_{=0}s + \underbrace{4B}_{=1}$$

$$\Leftrightarrow \quad B = \frac{1}{4}, \quad A = 0, \quad D = -\frac{1}{4}, \quad C = 0$$

$$\frac{1}{s^2(s^2+4)} = \frac{1}{4}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s^2+4}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

Table 6.1
$\omega = 2$

$$= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{4}\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= \frac{1}{4}t - \frac{1}{8}\sin(2t)$$

Using this, and the second shifting property:

$$\mathcal{L}^{-1}\left\{e^{-\pi s}\frac{1}{s^2(s^2+4)}\right\} = \left[\frac{1}{4}(t-\pi) - \frac{1}{8}\sin(2(t-\pi))\right]u(t-\pi)$$

Solution of IVP (written in terms of Heaviside function):

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} - \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{1}{s^2(s^2+4)}\right\}$$

$$x(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$
$$-\left[\frac{1}{4}(t-\pi) - \frac{1}{8}\sin(2(t-\pi))\right]\underbrace{u(t-\pi)}_{=0 \text{ or } 1}$$

(written as a piecewised defined function)

$$x(t) = \begin{cases} \frac{1}{4}t - \frac{1}{8}\sin(2t) & \text{if } 0 \leq t < \pi \\ \frac{1}{4}t - \frac{1}{8}\sin(2t) - \frac{1}{4}(t-\pi) + \frac{1}{8}\sin(2(t-\pi)) & \text{if } t \geq \pi \end{cases}$$

(and simplified) $\sin(2(t-\pi)) = \sin(2t - 2\pi) = \sin(2t)$

$$x(t) = \begin{cases} \frac{1}{4}t - \frac{1}{8}\sin(2t) & \text{if } 0 \leq t < \pi \\ \frac{\pi}{4} & \text{if } t \geq \pi \end{cases}$$

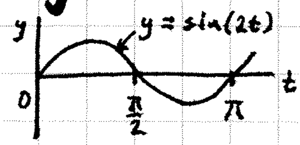
<u>Exercise</u> Show that $x''(t)$ is continuous, $x'''(t)$ is not (Compare: $f(t)$ is continuous, $f'(t)$ is not)

<u>Exercise</u> In Ex. 6.2.C, show that $x'(t)$ is continuous, $x''(t)$ is not (Compare: $f(t)$ is not continuous)

These two Exercises may give you some intuition into what properties to expect for a solution $x(t)$, of $Lx = f(t)$, when $f(t)$ is piecewise defined.

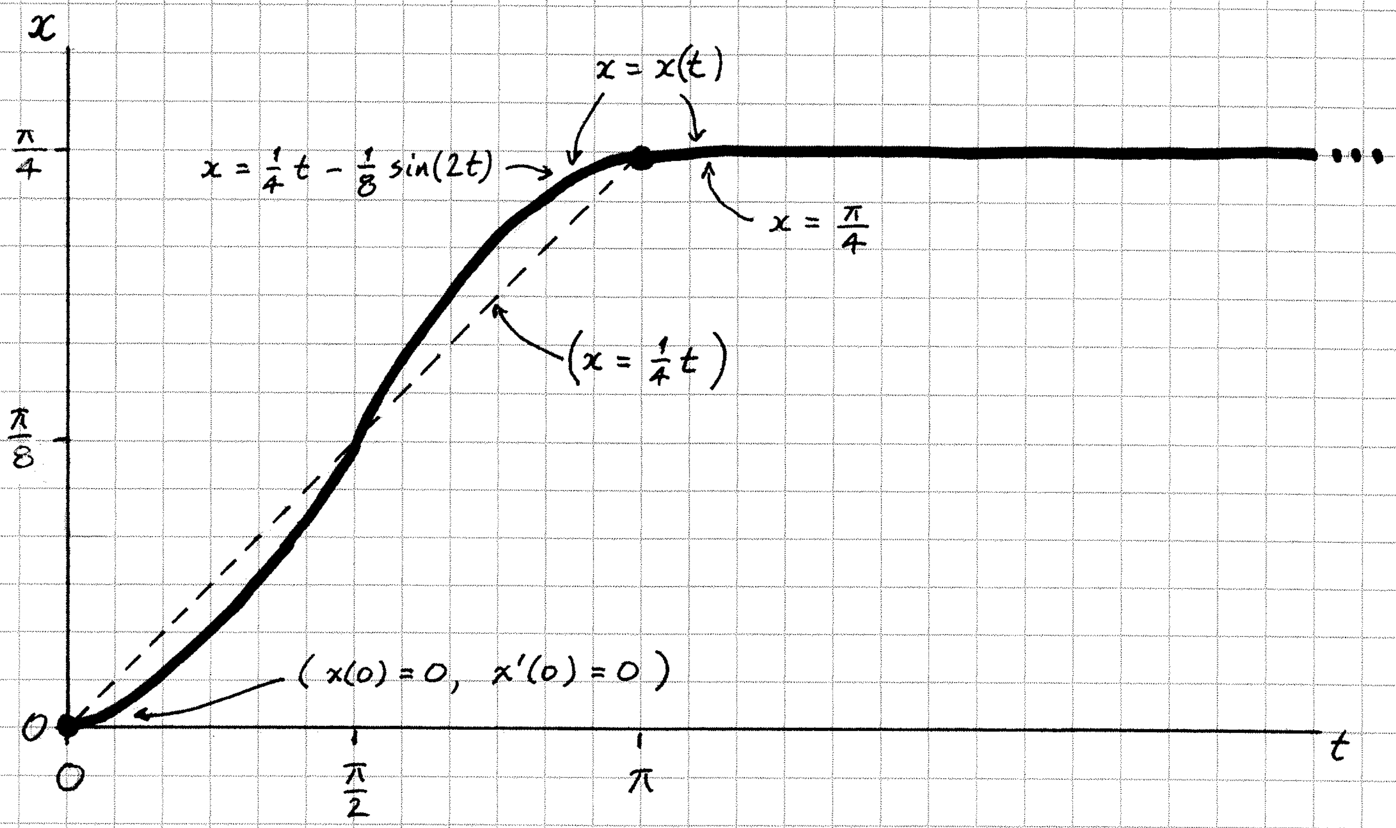
Graph of solution for $t \geq 0$: next page

Note that $\sin(2t)$ has a period of π:



and recall $x(0) = 0$, $x'(0) = 0$

This will help to draw the graph, without a calculator or $\underbrace{\text{Calc I methods}}_{\text{time-consuming}}$



Graph of solution $x(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t) - \left[\frac{1}{4}(t-\pi) - \frac{1}{8}\sin(2(t-\pi))\right]u(t-\pi)$

Note Scales on horizontal and vertical axes are different.