6.4 Dirac delta and impulse response

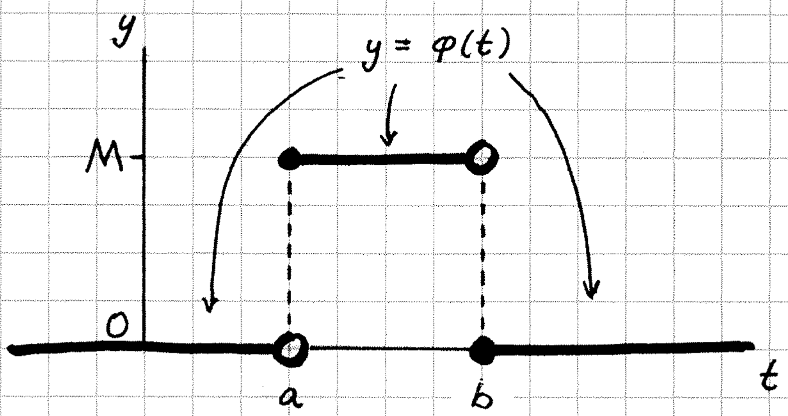
6.4.1 Rectangular pulse

Fix $a \geq 0$, let $b > a$.

Consider a constant external force of M newtons, applied to a mass-spring system, from $t = a$ until $t = b$, a rectangular pulse

$$\varphi(t) = \begin{cases} M & \text{if } a \leq t < b \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 0 & \text{if } t < a \\ M & \text{if } a \leq t < b \\ 0 & \text{if } t \geq b \end{cases}$$

$$= M\left[u(t-a) - u(t-b)\right]$$



Suppose $\int_0^\infty \varphi(t)\,dt = 1$ (*In physics this would be called the "impulse", with units* $N \cdot s = kg \cdot m \cdot s^{-1}$, *the same units as "momentum"* mv)

$$\int_0^\infty \varphi(t)\,dt = \int_0^a \underbrace{\varphi(t)}_{=0}dt + \int_a^b \underbrace{\varphi(t)}_{=M}dt + \int_b^\infty \underbrace{\varphi(t)}_{=0}dt$$

$$= 0 + M(b-a) + 0 = 1$$

$$\Rightarrow \quad M = \frac{1}{b-a}$$

$$\varphi(t) = \frac{1}{b-a}\left[u(t-a) - u(t-b)\right]$$

Take the Laplace transform:

$$\mathcal{L}\{\varphi(t)\} = \frac{1}{b-a}\left[\frac{e^{-as}}{s} - \frac{e^{-bs}}{s}\right]$$

$$= \frac{e^{-as} - e^{-bs}}{(b-a)s}$$

$$= e^{-as} \cdot \frac{1 - e^{-(b-a)s}}{(b-a)s}$$

Let $b = a+h$, $h>0$ (i.e. $h = b-a > 0$)

$$\mathcal{L}\{\varphi(t)\} = e^{-as} \cdot \frac{1 - e^{-sh}}{sh}$$

Exercise Find $\lim\limits_{h \to 0^+} \dfrac{1 - e^{-sh}}{sh}$ if $s > 0$.