HW5 due 11:59 pm Wed Apr 1 (34) Wed 2020-04-01

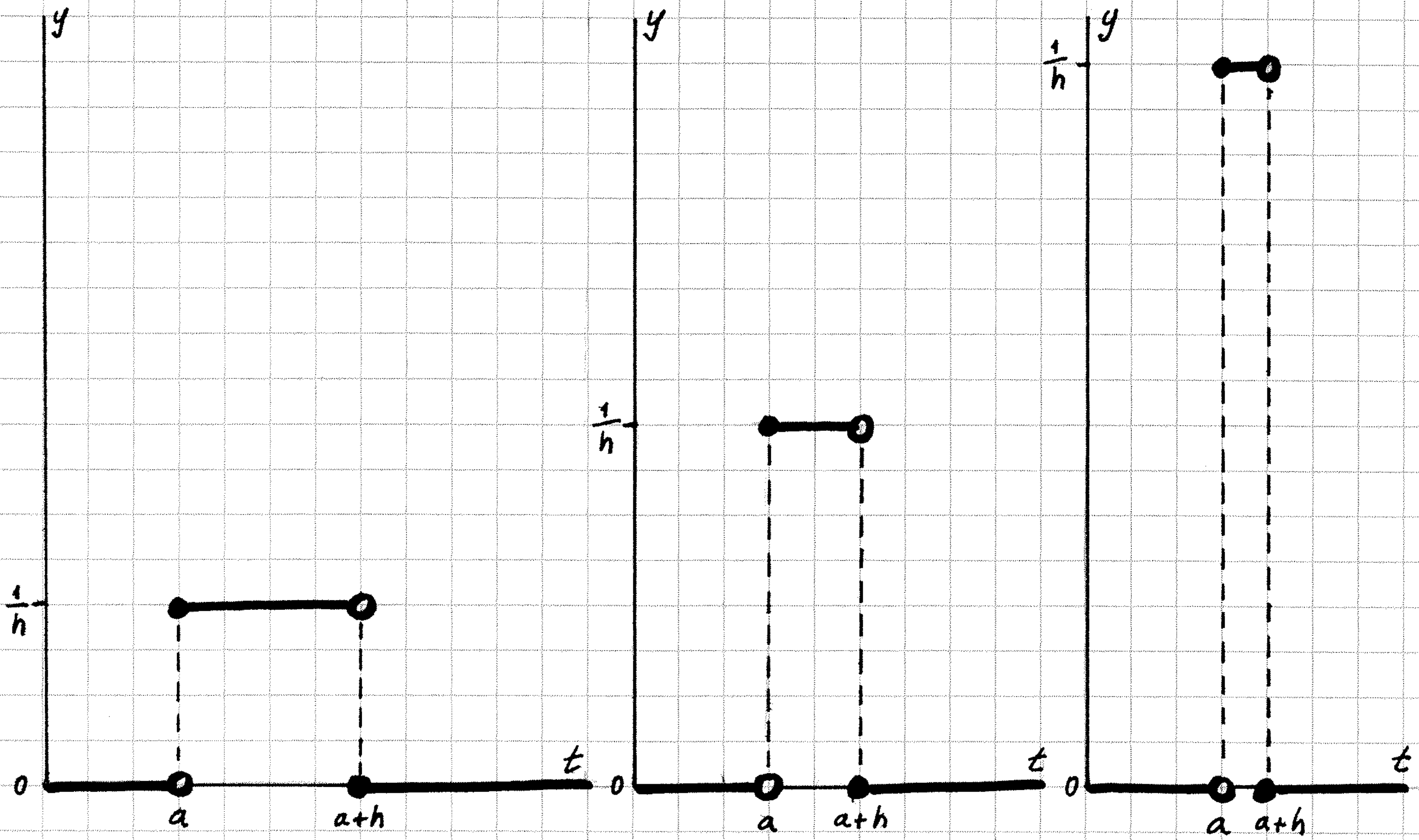
W5 Webwork due Wed Apr 8

6.4.2 The delta function

$$\varphi(t) = \frac{1}{b-a}\left[u(t-a) - u(t-b) \right] = \begin{cases} 0 & \text{if } t<a \\ \frac{1}{b-a} & \text{if } a \le t < b \\ 0 & \text{if } t \ge b \end{cases}$$

$$\mathcal{L}\{\varphi(t)\} = e^{-as}\,\frac{1-e^{-sh}}{sh} \quad \text{where } h = b-a.$$

If we let $h \to 0+$ then the rectangular pulse $\varphi(t)$ becomes narrower and higher, but keeps the same area:



Take the "limit" of $\varphi(t)$ as $h \to 0^+$: get a "function" $\delta(t-a)$ with its "area" of 1 concentrated at the single point $t=a$

$\delta(t-a)$ is called the **Dirac delta function**

but it is not a function in the sense of Calc I/II, it is a *generalized function*, or *distribution*

i. $\delta(t-a) = 0$ if $t \neq a$

ii. $$\int_c^d \delta(t-a)\,dt = \begin{cases} 1 & \text{if } c \leq a \leq d \\ 0 & \text{otherwise} \end{cases}$$

iii. $$\int_c^d \delta(t-a) f(t)\,dt = \begin{cases} f(a) & \text{if } c \leq a \leq d \\ 0 & \text{otherwise} \end{cases}$$

if $f(t)$ is continuous at $t=a$

(See p. 314 for a "proof" of iii.)

(In physics, a point mass of m kg located at position $x=a$ metres can be represented by a density function $m\,\delta(x-a)$.)

Sometimes it is useful to think of $\delta(t-a)$ as an "infinite" pulse at $t=a$ with width 0 and area 1.

Laplace transform:

$$\boxed{\mathcal{L}\{\delta(t-a)\} = e^{-as}} = \lim_{h \to 0^+} e^{-as} \frac{1-e^{-sh}}{sh}$$

$$\int_0^\infty e^{-st}\delta(t-a)\,dt = e^{-as} \quad \text{by iii.}$$

6.4.3 Impulse response

The **impulse response** for a (constant-coefficient, linear, 2nd-order) differential operator L is the solution to the IVP

$$\boxed{Lx = \delta(t), \quad x(0) = 0, \quad x'(0) = 0}$$

<u>**Example**</u> 6.4.A Find the impulse response for

$$L = \frac{d^2}{dt^2} + \omega_0^2$$

$$Lx = \left(\frac{d^2}{dt^2} + \omega_0^2\right)x = x'' + \omega_0^2 x = \delta(t), \quad x(0)=0, \; x'(0)=0$$

Let $X(s) = \mathcal{L}\{x(t)\}$

$$s^2 X(s) + \omega_0^2 X(s) = 1$$

$$X(s) = \frac{1}{s^2 + \omega_0^2}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega_0^2}\right\} = \frac{1}{\omega_0}\mathcal{L}^{-1}\left\{\frac{\omega_0}{s^2+\omega_0^2}\right\} = \frac{1}{\omega_0}\sin(\omega_0 t)$$