Example 6.4.B Solve $x'' - x = 3\delta(t-2)$, $x(0)=0$, $x'(0)=0$

Let $X(s) = \mathcal{L}\{x(t)\}$

$$s^2 X(s) - X(s) = 3e^{-2s}$$

$$X(s) = \frac{3e^{-2s}}{s^2-1} = 3e^{-2s}\frac{1}{s^2-1}$$

Partial fractions (Exercise):

$$\frac{1}{s^2-1} = \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}$$

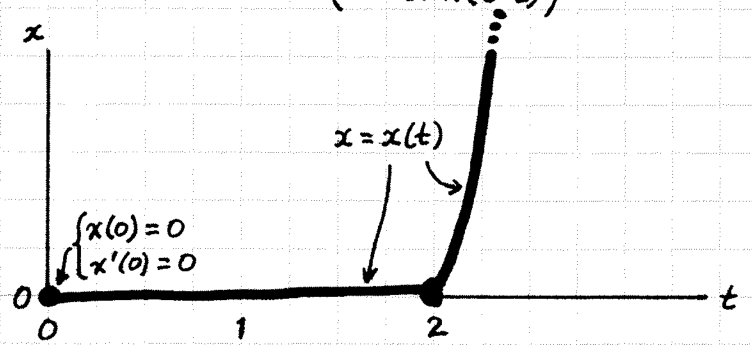
$$X(s) = \frac{3}{2}e^{-2s}\frac{1}{s-1} - \frac{3}{2}e^{-2s}\frac{1}{s+1}$$

$\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^{t}$

$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$

$$x(t) = \frac{3}{2}e^{t-2}u(t-2) - \frac{3}{2}e^{-(t-2)}u(t-2)$$

$$= 3\underbrace{\frac{e^{t-2} - e^{-(t-2)}}{2}}_{(\,=\,\sinh(t-2))}u(t-2) = \begin{cases} 0 & \text{if } t<2 \\ \sinh(t-2) & \text{if } t \geq 2 \end{cases}$$



"Nothing happens" until $t=2$

$x'(2^-) = 0$

$x'(2^+) = 3$

Jump discontinuity in x' at $t=2$ where the delta function is applied

Example 6.4.C Solve $x'' + \pi^2 x = f(t) + 2\delta(t-3)$,

$x(0)=0$, $x'(0)=0$, $f(t) = \begin{cases} 2 & \text{if } 1 \leq t < 2 \\ 0 & \text{otherwise} \end{cases}$

Let $X(s) = \mathcal{L}\{x(t)\}$

$$s^2 X(s) + \pi^2 X(s) = \underbrace{2e^{-s}\frac{1}{s} - 2e^{-2s}\frac{1}{s}}_{\mathcal{L}\{f(t)\}\ (\text{Exercise})} + \underbrace{2e^{-3s}}_{\mathcal{L}\{2\delta(t-3)\}}$$

$$X(s) = 2e^{-s}\frac{1}{s(s^2+\pi^2)} - 2e^{-2s}\frac{1}{s(s^2+\pi^2)} + 2e^{-3s}\frac{1}{s^2+\pi^2}$$

Partial fractions (Exercise): $\dfrac{1}{s(s^2+\pi^2)} = \dfrac{1}{\pi^2}\dfrac{1}{s} - \dfrac{1}{\pi^2}\dfrac{s}{s^2+\pi^2}$

$$X(s) = \frac{2}{\pi^2}e^{-s}\left[\frac{1}{s} - \frac{s}{s^2+\pi^2}\right] - \frac{2}{\pi^2}e^{-2s}\left[\frac{1}{s} - \frac{s}{s^2+\pi^2}\right] + \frac{2}{\pi}e^{-3s}\frac{\pi}{s^2+\pi^2}$$

$$x(t) = \frac{2}{\pi^2}\Big[1 - \underbrace{\cos(\pi(t-1))}_{-\cos(\pi t)}\Big]u(t-1) - \frac{2}{\pi^2}\Big[1 - \underbrace{\cos(\pi(t-2))}_{\cos(\pi t)}\Big]u(t-2)$$
$$+ \frac{2}{\pi}\underbrace{\sin(\pi(t-3))}_{-\sin(\pi t)}u(t-3)$$

$$x(t) = \begin{cases} 0 & \text{if } t < 1 \\ \frac{2}{\pi^2}\left[1 + \cos(\pi t)\right] & \text{if } 1 \le t < 2 \\ \frac{4}{\pi^2}\cos(\pi t) & \text{if } 2 \le t < 3 \\ \frac{4}{\pi^2}\cos(\pi t) - \frac{2}{\pi}\sin(\pi t) & \text{if } t \geq 3 \end{cases}$$