Vibrating string (p.11), the wave equation (p.33)

Homework Assignment 1 due Fri Sep 25

[Last lecture: Second order PDEs, classification of linear (and semilinear) equations]

3. The Wave and Diffusion (Heat) Equations

Vibrating string (deriving the wave equation)

Some more modelling. Consider a flexible elastic homogeneous (uniform) string (or thread or wire etc.), tightly stretched along the $x$-axis, moving so the string stays in a plane: assume all movements are transverse (e.g. vertical) to the $x$-axis (e.g. on a musical instrument). Let

- $x$ [m] distance along string in its equilibrium position
- $t$ [s] time
- $u$ [m] transverse displacement from equilibrium
- $\rho$ [kg·m$^{-1}$] constant (linear) density of homogeneous string

Assume movements are small: so that we can approximate the movement as being only transverse. Then the transverse velocity is
An infinitesimally small element $dx$ of string can be treated as a point with infinitesimal mass $\rho \, dx$ and transverse momentum $\rho \, u_t \, dx$. In any arbitrary interval of string $a \leq x \leq b$ (i.e. $x \in [a, b]$) at any time $t$ we have

Newton’s Second Law of mechanics is conservation of momentum

Assume the string is *flexible* and *elastic*: it does not resist deformation so the tension (force) $T(x, t)$ is always directed tangentially to the string. We balance the transverse forces on the interval $[a, b]$ of string

$$
\int_a^b \left\{ \rho \, u_{tt}(x, t) - \frac{\partial}{\partial x} \left[ T(x, t) \, \frac{u_x(x, t)}{\sqrt{1 + (u_x(x, t))^2}} \right] \right\} \, dx = 0
$$
Since this is true for any \( t \), for any closed interval \([a, b]\), we must have

or

The assumption of small vibrations also implies

so we make the approximation

We also assume the string is \textit{tightly stretched}, which corresponds mathematically to

so we get

where \( T > 0 \) and \( \rho > 0 \) are constants. Equivalently,

\[
\begin{align*}
\frac{d^2 u}{dt^2} - c^2 \frac{d^2 u}{dx^2} = 0
\end{align*}
\]  \hspace{1cm} (1.3.2)

where \( c = \sqrt{T/\rho} > 0 \) is a constant. Equation (1.3.2) is called the (homogeneous) \textbf{wave equation}. 
The wave equation (general solution)

The wave equation (1.3.2) is hyperbolic (for all $x, t$):

If we make a coordinate change $(x, t) \rightarrow (\xi, \eta)$ we can derive equations for $\xi(x, t), \eta(x, t)$ by letting

$$u(x, t) = U(\xi, \eta)$$

and calculating, using the multivariable Chain Rule:
and then

$\mathbf{−c^2_ξxη_x + ξ_tη_t = −c^2(1)(1) + (c)(−c) = −2c^2}$

so

$\mathbf{−c^2_ξxη_x + ξ_tη_t = −c^2(1)(1) + (c)(−c) = −2c^2}$

and the PDE that is equivalent to $\mathbf{u_{tt} − c^2 u_{xx} = 0}$ in the new coordinates $(\xi, \eta)$ is $\mathbf{−4c^2U_{ξη} = 0}$,
or

\[ U_{\xi\eta} = 0 \]

Write this as

\[ (U_\xi)_\eta = \frac{\partial}{\partial \eta} U_\xi(\xi, \eta) = 0 \]

which has the general solution

\[ U_\xi(\xi, \eta) = F(\xi) \]

some arbitrary function of \( \xi \) only, then

\[ U_\xi(\xi, \eta) = \frac{\partial}{\partial \xi} U(\xi, \eta) = F(\xi) \]

has the general solution

\[ U(\xi, \eta) = \]

i.e.

\[ U(\xi, \eta) = \]

where \( f \) and \( g \) are arbitrary functions. Now remember \( u(x, t) = U(\xi, \eta) \)
so substituting \( \xi = x + ct, \eta = x - ct \) we get the general solution of the homogeneous wave equation (1.3.2)

\[ u(x, t) = \]

(2.1.3)

where \( f \) and \( g \) are arbitrary functions.