MATH 552 (2023W1) Lecture 1: Wed Sep 06

# INTRODUCTION TO DYNAMICAL SYSTEMS

## 1. Linear Dynamical Systems

Review of homogeneous linear systems of ODEs

Please read at least the first two pages of Appendix **A. Review of (Multivariable) Differential Calculus**.

Let $\mathcal{J} = (-\delta_1, \delta_2)$, where $-\infty \leq -\delta_1 < \delta_2 \leq +\infty$, be an open interval and let $A(t)$ be a real $n \times n$ matrix of coefficient functions that are all continuous on $t \in \mathcal{J}$. Then according to a basic theorem of ODEs, the homogeneous linear system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $\dot{} = \frac{d}{dt}$, $x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}^\mathsf{T} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, always has $n$ linearly independent real (continuous, $n$-vector) solutions on $\mathcal{J}$, (for more details, see almost any undergraduate ODE textbook). For example, if $n = 2$, then (1.1) can be written more explicitly as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

and any 2 linearly independent solutions have the form

$$\psi^{[1]}(t) = \begin{pmatrix} \psi_1^{[1]}(t) \\ \psi_2^{[1]}(t) \end{pmatrix}, \ \psi^{[2]}(t) = \begin{pmatrix} \psi_1^{[2]}(t) \\ \psi_2^{[2]}(t) \end{pmatrix}.$$

Putting $n$ linearly independent solutions of (1.1) as the columns of a real $n \times n$ matrix, we get a **fundamental matrix**

A fundamental matrix (it is not unique – why not?) satisfies

$$\dot{\Psi} = A(t)\Psi, \quad \det \Psi(t) \neq 0 \quad \text{for all } t \in \mathcal{J}, \tag{1.2}$$

and a general solution for (1.1) can be written in the form

$$x(t) = \Psi(t)c, \quad c = (\ c_1 \ \cdots \ c_n\ )^{\mathsf{T}} \in \mathbb{R}^n \text{ arbitrary.} \tag{1.3}$$

**Example 1.A./Exercise.** $\dot{x} = A(t)x$, $x \in \mathbb{R}^2$, where

$$A(t) = \begin{pmatrix} -1 + \frac{3}{2}\cos^2(t) & 1 - \frac{3}{2}\sin(t)\cos(t) \\ -1 - \frac{3}{2}\sin(t)\cos(t) & -1 + \frac{3}{2}\sin^2(t) \end{pmatrix} \quad t \in \mathbb{R}.$$

Verify that

is a fundamental set of solutions. Find a general solution, form a fundamental matrix, and check (1.2)–(1.3) explicitly. Then prove (1.2)–(1.3) for a general (1.1).

If $A(\cdot)$ is continuous on $\mathcal{J}$ and if $t_0 \in \mathcal{J}$, then another basic theorem of ODEs states that, for any $x_0 \in \mathbb{R}^n$, the initial value problem

$$\dot{x} = A(t)x, \quad x(t_0) = x_0 \tag{1.4}$$

has a unique solution which we denote

$$x(t) = \varphi(t, t_0, x_0) \in \mathbb{R}^n,$$

for $t \in \mathcal{J}$. For a homogeneous linear system, this unique solution of (1.4) can be expressed in terms of a fundamental matrix $\Psi(t)$ as (**Exercise**)

$$\varphi(t, t_0, x_0) = \Psi(t)\Psi(t_0)^{-1}x_0. \tag{1.5}$$

For an initial time $t_0 \in \mathcal{J}$, **the principal (fundamental) matrix at** $t_0$ is the unique fundamental matrix $\Psi(t) = M(t, t_0)$ that solves the matrix initial value problem

$$\dot{\Psi} = A(t)\Psi, \quad \Psi(t_0) = I_n,$$

where $I_n$ denotes the $n \times n$ identity matrix. If $x_0 \in \mathbb{R}^n$ is given, then the unique solution $x(t) = \varphi(t, t_0, x_0)$ of the initial value problem

$$\dot{x} = A(t)x, \quad x(t_0) = x_0 \in \mathbb{R}^n,$$

is

$$x(t) = \varphi(t, t_0, x_0) = M(t, t_0)x_0.$$

By the above discussion, we always have

$$M(t, t_0) = \Psi(t)\Psi(t_0)^{-1}$$

(although this may not always be the most convenient way to find $M(t, t_0)$).

For $t_0 \in \mathcal{J}$ and $x_0 \in \mathbb{R}^n$, the **solution curve** passing through the point $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ is

$$Cr(t_0, x_0) = \{(t, x) : \ x = \varphi(t, t_0, x_0), \ t \in \mathcal{J}\} \subseteq \mathbb{R} \times \mathbb{R}^n.$$

**Example 1.B.** $\dot{x} = [-\frac{1}{2} + \cos(t)]\, x,\ \ x(0) = 1 \in \mathbb{R}^1.$

**Example 1.C.** $\dot{x} = -\frac{1}{2}\, x,\ \ x(0) = 1 \in \mathbb{R}^1.$

<u>Linear flows, linear continuous-time dynamical systems</u>

If $A$ is a constant real $n \times n$ matrix, then the (constant-coefficient) homogeneous linear system of ODEs

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n \tag{1.6}$$

is called **autonomous**, $\mathcal{J} = \mathbb{R}$, and the right-hand side $Ax$ of the ODE is called a **linear vector field** in this case where it does not depend explicitly on $t$.

**Exercise.** Let $x(t) = \varphi(t, t_0, x_0)$ denote the unique solution of the initial value problem $\dot{x} = Ax$, $x(t_0) = x_0$, and let $y(t) = \varphi(t, 0, x_0)$ denote the unique solution of the initial value problem $\dot{y} = Ay$, $y(0) = x_0$. Show that $x(t) = y(t - t_0)$.

Thus without loss of generality, we may always take the initial time as $t_0 = 0$ in an initial value problem for an autonomous system, and consider

$$\dot{x} = Ax, \quad x(0) = x_0 \in \mathbb{R}^n. \tag{1.7}$$

For autonomous homogeneous linear systems, the principal matrix $M(t, 0)$ is called the **linear flow** (or **linear evolution operator**).

The **exponential matrix** $e^{At}$ is defined to be

(If you are worried about convergence, you can prove as an exercise that the series is absolutely convergent in matrix norm, for every $t \in \mathbb{R}$.)

**Theorem 1.1** (Linear flow)**.**

$$M(t,0) = e^{At}.$$

The triple $\{\mathbb{R}, \mathbb{R}^n, e^{At}\}$ is a **linear continuous-time dynamical system**, where $\mathbb{R}$ is the **time set**, $\mathbb{R}^n$ is the **state space** (or **phase space**), $\{e^{At}\}_{t\in\mathbb{R}}$ is the **family of linear evolution operators**, or the **linear flow**. Often, for brevity we will refer to a linear continuous-time dynamical system as a linear flow.

A linear flow satisfies

$$= \qquad \text{(DS.0)}$$

$$= \qquad \text{(DS.1)}$$

for all $s$, $t \in \mathbb{R}$. The first property (DS.0) follows directly from the definition of the exponential matrix, the second (DS.1) follows from a result in Homework Assignment 1.