MATH 552 (2023W1) Lecture 2: Fri Sep 08

[**Last lecture:** Homogeneous linear (systems of) ODEs $\dot{x} = A(t)x$. **Autonomous** homogeneous linear ODEs $\dot{x} = Ax$: generate (by solving initial value problems for all possible initial values) linear flows $\{e^{At}\}_{t\in\mathbb{R}}$.]

Geometry of linear flows

For a point x_0 in state space $\mathbb{R}^n$, its **orbit** (or **trajectory**) is the subset of state space

oriented by the direction of increasing t. The **phase portrait** of a linear flow is a partitioning of the state space into orbits. Notice that orbits are 1-to-1 projections of solution curves onto the state space (guaranteed, for autonomous systems).

We look at some basic examples with $n = 2$: consider $\dot{x} = Ax$, $x \in \mathbb{R}^2$.

Example 1.D./Exercise. (Real, and Jordan, normal form for 2 distict real eigenvalues)

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \text{where } \lambda_1,\ \lambda_2 \in \mathbb{R},\ \lambda_1 \neq \lambda_2.$$

(a) Solve the matrix initial value problem $\dot{\Psi} = A\,\Psi$, $\Psi(0) = I_2$ (whose solution is defined to be $\Psi(t) = M(t, 0)$), in other words, find the 2 solutions, $x = \psi^{[1]}(t)$ of $\dot{x} = Ax$, $x(0) = (\ 1\ \ 0\ )^{\mathsf{T}}$ and $x = \psi^{[2]}(t)$ of $\dot{x} = Ax$, $x(0) = (\ 0\ \ 1\ )^{\mathsf{T}}$, then form the matrix $\Psi(t) = M(t, 0)$, to find

$$M(t, 0) = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}.$$

(b) Use the matrix series definition for e^{At} and sum all resulting component series to check that

so we have a simple example that illustrates Theorem 1.1.

Phase portrait, if (i) $\lambda_1 < 0 < \lambda_2$ (the origin is an "unstable saddle"):

Draw your own phase portraits if (ii) $\lambda_1 < \lambda_2 < 0$ (the origin is a "stable node"); (iii) $0 < \lambda_1 < \lambda_2$ (the origin is an "unstable node"); (iv) $\lambda_1 < 0 = \lambda_2$; (v) $\lambda_1 = 0 < \lambda_2$.

Example 1.E./Exercise. (Real normal form for 2 nonreal complex conjugate eigenvalues)

$$A = \begin{pmatrix} \mu_1 & -\omega_1 \\ \omega_1 & \mu_1 \end{pmatrix}, \quad \text{where } \mu_1,\, \omega_1 \in \mathbb{R} \text{ and } \omega_1 > 0.$$

(a) Verify that the eigenvalues of A are the nonreal, complex conjugate numbers $\lambda_1 = \mu_1 + i\omega_1$, $\lambda_2 = \mu_1 - i\omega_1 = \bar{\lambda}_1$.

(b) Solve the matrix initial value problem $\dot{\Psi} = A\Psi$, $\Psi(0) = I_2$, to find $\Psi(t) = M(t, 0)$.

(c) Use the matrix series definition for e^{At} and sum all resulting component series and check that

where $M(t, 0)$ is calculated in part (b).

(d) Because the eigenvalues are nonreal, it is worth making an excursion into complex coordinates, before ultimately coming back to the original, real coordinates to calculate $M(t, 0)$ for a third time. **Complexify** $\dot{x} = Ax$ by thinking of $x \in \mathbb{C}^2$ (with the same real matrix A). Then make a linear nonsingular coordinate change in $\mathbb{C}^2$

i.e. $x = P\,z$ or

$$z = \begin{pmatrix} z_1 \\ \\ z_2 \end{pmatrix} = P^{-1} \begin{pmatrix} x_1 \\ \\ x_2 \end{pmatrix}, \quad \text{where } P^{-1} = \begin{pmatrix} \qquad\qquad & \\ & \qquad\qquad \end{pmatrix}$$

(note that $z_2 = \bar{z}_1$ if and only if the complex numbers x_1 and x_2 are both purely real). Find the matrix P. Transform the ODE to the new coordinates

and verify that

$$\dot{z} = Jz, \quad \text{where } J = P^{-1}AP = \begin{pmatrix} \qquad\qquad & \\ & \qquad\qquad \end{pmatrix}$$

is now a diagonal, but nonreal, matrix (this is the Jordan normal form of the matrix in this case). Proceeding formally, we easily "solve" the matrix initial value problem $\dot{\Phi} = J\Phi$, $\Phi(0) = I_2$ just as in Example 2.D (a), to get

$$\Phi(t) = \begin{pmatrix} \qquad\qquad & \\ & \qquad\qquad \end{pmatrix} = e^{Jt}$$

and the "solution" of the initial value problem $\dot{z} = Jz$, $z(0) = z_0$, is $z(t) = e^{Jt}z_0$, with

Verify that if $z_2(0) = \bar{z}_1(0)$, then $z_2(t) = \bar{z}_1(t)$ for all $t \in \mathbb{R}$ (we say that the "real" subspace of $\mathbb{C}^2$, $\{z_2 = \bar{z}_1\} \cong \mathbb{R}^2$, is dynamically **invariant**). Now "**realify**" the "solution" by restricting it to the dynamically invariant real subspace. Transform back to the x-coordinates, which are now real, and verify that you get

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}.$$

Verify that the matrix

$$\begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

is the same as $M(t, 0)$ found in part (b), and e^{At} found in part (c). (The simplest way to justify this complexify/realify procedure, without worrying about the theory of complex dynamical systems, is just to check that you get a correct real solution in the end).

(e) Another way to solve the initial value problem (for the fourth time) is to use the familiar **polar coordinates** for $x \in \mathbb{R}^2 \setminus \{0\}$,

$$x_1 = r\ \cos(\theta), \quad x_2 = r\ \sin(\theta),$$

so that

$$r = \|x\| = \sqrt{x_1^2 + x_2^2} > 0, \quad \tan(\theta) = \frac{x_2}{x_1}\ (\text{if } x_1 \neq 0)$$

Verify that under this (now nonlinear) change of coordinates (use the chain rule) that $\dot{x} = Ax$ becomes

The initial value problems for these ODEs are easily solved, to get

But, we must remember that adding any integer multiple of 2π to θ corresponds to the same point x in $\mathbb{R}^2 \setminus \{0\}$. To take this into account, we think of the argument, or polar angle, θ as belonging *not* to $\mathbb{R}^1$, but *instead* to **the circle** $\mathbb{S}^1$

$$\theta \in \mathbb{S}^1$$

where the circle is defined as

$$\mathbb{S}^1 = \mathbb{R}^1 / 2\pi\mathbb{Z}^1$$

and this notation means $\theta_2 \in \mathbb{R}^1$ is *identified with* (or is considered *equivalent to*) $\theta_1 \in \mathbb{R}^1$ if and only if $\theta_2 - \theta_1 \in 2\pi\mathbb{Z}^1$ (i.e. θ_2 and θ_1 differ by an integer multiple of 2π). To indicate that we are thinking of $\theta(t)$ as belonging to the circle and not the real line, we write

A typical orbit is represented for $(r, \theta) \in \mathbb{R}_+ \times \mathbb{S}^1$, if (i) $\mu_1 > 0$, as

while the same orbit in rectangular coordinates $x \in \mathbb{R}^2$ looks like

(the origin is an "unstable focus"). Draw both representations of a typical orbit (with $\omega_1 > 0$), if (ii) $\mu_1 < 0$ (the origin is a "stable focus"), and if (iii) $\mu_1 = 0$ (the origin is a "centre"). What if $\omega_1 < 0$?

Later, when we study the Hopf bifurcation, we will be flipping back and forth among the three different coordinate systems for the plane: real rectangular, complexified rectangular, real polar.

Please read the first 5 pages of Appendix **B. Some Linear Algebra**.