MATH 552 (2023W1) Lecture 3: Mon Sep 11

[**Last lecture:** Orbits and phase portraits, Two examples of $\dot{x} = Ax$, $x \in \mathbb{R}^2$, $\{e^{At}\}_{t\in\mathbb{R}}$, with A having (Example 1.B) real distinct eigenvalues, (Example 1.C) nonreal complex conjugate eigenvalues.]

A third important example – multiple eigenvalues:

Example 1.F./Exercise. (Real, and Jordan, normal forms for a real eigenvalue of multiplicity 2)

Case i. ("nongeneric" case)

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \quad \text{where } \lambda_1 \in \mathbb{R},$$

Case ii. ("generic" case)

$$A = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}, \quad \text{where } \lambda_1 \in \mathbb{R},$$

(a) In both cases, the characteristic polynomial of A is

$$h(\lambda) = \det(A - \lambda I_2) = (\lambda_1 - \lambda)^2.$$

λ_1 is a root of $h(\lambda)$ (i.e. an eigenvalue of A) of (algebraic) multiplicity $m_1 = 2$. Verify that in case i, there are two linearly independent eigenvectors for λ_1, but in case ii, there is only one linearly independent eigenvector

$v_1^{[1]} \in \mathbb{R}^2$ for λ_1 (i.e. in case ii the *kernel*, or *nullspace* $\mathcal{N}(A - \lambda_1 I_2)$ has dimension $1 < m_1$ and $\mathcal{N}(A - \lambda_1 I_2) = \text{span}\{v_1^{[1]}\}$).

(b) Case i is similar to Example 1.D, 2 distict real eigenvalues. In case ii, solve the matrix initial value problem $\dot{\Psi} = A\Psi$, $\Psi(0) = I_2$, whose solution is defined to be $\Psi(t) = M(t, 0)$ to find the principal matrix $M(t, 0)$.

(c) Notice that in case ii, we have $A = S + N$, where

$$S = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}, N = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

S is **semisimple** (diagonalizable), N is **nilpotent** (some postitive integer power is zero). Check that $SN = NS$, then we can use a result from Homework Assignment 1 to get $e^{(S+N)t} = e^{St}e^{Nt}$. Compute e^{St} (see Example 1.B), and use the series definition to compute e^{Nt} (this series terminates in a finite sum because N is nilpotent). Then verify in case ii,

$$e^{At} = s^{St}e^{Nt} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} = M(t, 0).$$

(d) From part (a), $v_1^{[1]} = (\ 1\ \ 0\ )^\mathsf{T}$ will work as an eigenvector in both cases. In case ii, $v_1^{[2]} = (\ 0\ \ 1\ )^\mathsf{T}$ is not an eigenvector. But, check that in case ii, $v_1^{[2]}$ satisfies and thus $(A - \lambda_1 I_2)^2 v_1^{[2]} = 0$, and in fact in case ii the generalized eigenspace for the real eigenvalue λ_1 is

$$X(\lambda_1) = \mathcal{N}((A - \lambda_1 I_2)^{m_1}) \cap \mathbb{R}^2 = \text{span}\{v_1^{[1]}, v_1^{[2]}\}.$$

(e) For case ii, draw the phase portrait if (i) $\lambda_1 < 0$; (ii) > 0; (iii) $= 0$.

The real normal form of a matrix

It is possible to find the linear flow e^{At} explicitly, by first finding a basis of (possibly complex) **generalized eigenvectors** of A and then using these to determine a *real* nonsingular linear coordinate change T that takes the real matrix A into its **real normal form** R. The linear flow e^{Rt} for a linear vector field with the matrix R in real normal form can be computed explicitly without too much trouble (for examples, see Homework Assignment 1).

See Appendix **B. Linear Algebra**. Here in these lectures we give a summary and an example. The **sum** of subspaces V_1 and V_2 is

and, if furthermore $V_1 \cap V_2 = \{0\}$, then the sum is called a **direct sum**

If $V = V_1 \oplus V_2$, then every $v \in V$ has a *unique* decomposition as

By factoring the characteristic polynomial $h(\lambda) = \det(A - \lambda I_n)$ completely over $\mathbb{C}$ into linear factors,

$$h(\lambda) = (\lambda_1 - \lambda)^{m_1} \cdots (\lambda_d - \lambda)^{m_d},$$

we find all the distinct eigenvalues $\lambda_1, \ldots, \lambda_d$ and their (algebraic) multiplicities $m_1, \ldots, m_d$. An important theorem in linear algebra states that

$$\mathbb{C}^n = \mathcal{N}((A - \lambda_1 I_n)^{m_1}) \oplus \cdots \oplus \mathcal{N}((A - \lambda_d I_n)^{m_d}),$$

where each algebraically invariant subspace $\mathcal{N}((A - \lambda_j I_n)^{m_j})$ of $\mathbb{C}^n$ is spanned by a basis of m_j vectors, called **generalized eigenvectors**:

$$\mathcal{N}((A - \lambda_j I_n)^{m_j}) = \operatorname{span}\{v_j^{[1]}, \ldots, v_j^{[m_j]}\}.$$

If an eigenvalue $\lambda_j \in \mathbb{C}$ happens to be *real* (i.e. $\operatorname{Im}(\lambda_j) = 0$) then it is possible to choose the basis of generalized eigenvectors $v_j^{[1]}, \ldots, v_j^{[m_j]}$ so that they are all real, and we will assume this has been done, so we can think of each $v_j^{[k]} \in \mathbb{R}^n$.

"there exists a direct sum of algebraically invariant subspaces" $\Leftrightarrow$

"the matrix can be block-diagonalized with respect to some basis"

By arranging the n generalized eigenvectors $v_j^{[k]} \in \mathbb{C}^n$ to be columns of a nonsingular $n \times n$ matrix P, the linear change of variables

$$x = Pz, \quad x, z \in \mathbb{C}^n$$

transforms the complexified $\dot{x} = Ax$, $x \in \mathbb{C}^n$ into the equivalent

$$\dot{z} = Jz, \quad z \in \mathbb{C}^n,$$

where $J = P^{-1}AP$ is the block-diagonal **Jordan normal form** of A (see **Theorem 1.2**, in Appendix B).

Theorem 1.2 (Jordan Normal Form)**.** *Let A be a real $n \times n$ matrix. Then there exists a linear nonsingular change of variables $x = Pz$ (if all the eigenvalues of A are real, then the matrix P can be chosen to be real; if A has any nonreal eigenvalue, then P is nonreal) that transforms*

$$\dot{x} = Ax \qquad \text{into} \qquad \dot{z} = Jz,$$

where $J = P^{-1}AP$ is block-diagonal, with square blocks of various sizes along the diagonal

$$J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_L \end{pmatrix}$$

and zeros elsewhere. Each block J_ℓ, $\ell = 1, \cdots, L$ has the form

$$J_\ell = (\, \lambda_j \,) \qquad \text{or} \qquad J_\ell = \begin{pmatrix} \lambda_j & 1 & & \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{pmatrix},$$

i.e. a 1×1 block containing an eigenvalue, or a larger block with the eigenvalue λ_j in each position on the main diagonal, the number 1 in each position on the first superdiagonal, and 0 in every other position.

E.g. suppose $n = 11$ and

$$h(\lambda) = (-5-\lambda)(-3-\lambda)^4(2i-\lambda)(-2i-\lambda)(1+3i-\lambda)^2(1-3i-\lambda)^2,$$

then we *could* have

$$J = \begin{pmatrix} -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+3i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+3i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-3i & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-3i \end{pmatrix}$$

There are 6 distinct eigenvalues and 7 elementary Jordan blocks, which shows that the number of distinct eigenvalues is not always the same as the number of elementary Jordan blocks.

If A is real, and if λ_j is a nonreal (i.e. $\mathrm{Im}(\lambda_j) \neq 0$) eigenvalue with multiplicity m_j, then the complex conjugate $\bar{\lambda}_j$ is also an eigenvalue with the same multiplicity. In this case the change of coordinates matrix P is nonreal, and generalized eigenvectors for λ_j, $\bar{\lambda}_j$ can (and should) be chosen in complex conjugate pairs

$$v_j^{[k]} = a_j^{[k]} + i b_j^{[k]}, \quad \bar{v}_j^{[k]} = a_j^{[k]} - i b_j^{[k]},$$

where $a_j^{[k]}$ and $b_j^{[k]}$ are *real* and linearly independent vectors.

To form a *real* change of basis matrix T from P: (a) for each *real* eigenvalue λ_j choose the same *real* generalized eigenvectors $v_j^{[k]}$ as were used to form P, to be columns of T; (b) for each *nonreal* eigenvalue λ_j with $\mathrm{Im}(\lambda_j) > 0$ choose the *real* vectors

$$a_j^{[k]}, \quad -b_j^{[k]}$$

to be columns of the matrix T. *NOTICE THE MINUS SIGN!*

Then the *real* change of variables

$$x = Ty, \quad x, y \in \mathbb{R}^n$$

transforms the real vector field $\dot{x} = Ax$, $x \in \mathbb{R}^n$ into the real vector field

$$\dot{y} = Ry, \quad y \in \mathbb{R}^n,$$

where $R = T^{-1}AT$ is the **real normal form** of A (see **Theorem 1.3**, in Appendix B).

Theorem 1.3 (Real Normal Form)**.** *Let A be a real $n \times n$ matrix. Then there is always a real linear nonsingular change of variables $x = Ty$ that transforms*

$$\dot{x} = Ax \qquad \textit{into} \qquad \dot{y} = Ry,$$

where $R = T^{-1}AT$ is block diagonal with square blocks of various sizes along the diagonal

$$R = \begin{pmatrix} R_1 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_M \end{pmatrix}$$

and zeros elsewhere.

i) If λ_j is a real eigenvalue, then a corresponding block R_m $(m = 1, \cdots, M)$ is an elementary Jordan block

$$R_m = (\, \lambda_j \,) \qquad \textit{or} \qquad R_m = \begin{pmatrix} \lambda_j & 1 & & \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{pmatrix};$$

ii) If $\lambda_j = \mu_j + i\omega_j$ (μ_j and ω_j both real, $\omega_j > 0$) is a nonreal eigenvalue, then a corresponding block R_m $(m = 1, \cdots, M)$ has the form

$$R_m = D_j \qquad \textit{or} \qquad R_m = \begin{pmatrix} D_j & I_2 & & \\ & D_j & \ddots & \\ & & \ddots & I_2 \\ & & & D_j \end{pmatrix},$$

where D_j and I_2 are the 2×2 matrices

$$D_j = \begin{pmatrix} \mu_j & -\omega_j \\ \omega_j & \mu_j \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

E.g. the real normal form corresponding to the previous Jordan normal form is

$$R = \begin{pmatrix}
-5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1
\end{pmatrix}$$