MATH 552 (2022W1) Lecture 4: Wed Sep 13

[**Last lecture:** Example (1.F) with double eigenvalues (especially the "generic" case), Jordan normal forms, real normal forms]

Invariant sets, hyperbolicity and stability for linear flows

For a linear flow in $\mathbb{R}^n$, a subset $S \subseteq \mathbb{R}^n$ is a (dynamically)

(a) **invariant** set,

(b) **positively invariant** set,

(c) **negatively invariant** set, or

(d) **locally invariant** set,

if an initial value $x(0) = x_0 \in S$ implies the solution of the initial value problem $x(t) = \varphi(t, 0, x_0) \in S$ (i.e. $\varphi(t, 0, S) \subseteq S$) for all

(a)

(b)

(c)

(d)

respectively.

Thus, invariant sets consist entirely of orbits (or parts of orbits). An **invariant subspace** is an algebraic subspace that is (dynamically) invariant. Generalized eigenspaces are both algebraically invariant and (dynamically) invariant. From now on we will often omit the adjective "dynamically" when talking about invariant sets.

If A is a constant real $n \times n$ matrix, then qualitative properties of the linear flow e^{At}, generated by the linear vector field Ax through the linear autonomous ODE

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n,$$

are determined in "most" cases by the real parts $\mathrm{Re}(\lambda_j)$ of the eigenvalues λ_j of the matrix of coefficients A.

Exercise. Show that if $\mathbb{R}^n = V_1 \oplus V_2$ is the direct sum decomposition of $\mathbb{R}^n$ into subspaces V_1 and V_2 that are algebraically invariant under a real $n \times n$ matrix A (i.e. $AV_j \subseteq V_j$, $j = 1, 2$), then V_1 and V_2 are invariant subspaces (i.e. with respect to the linear vector field $\dot{x} = Ax$, or the linear flow e^{At}). *Hint:* With respect to some suitable basis, the matrix representing A is block diagonal.

Stable, unstable and centre subspaces: For any real $n \times n$ matrix A:
(a) the **stable subspace** T^s is the direct sum of the generalized eigenspaces corresponding to all the eigenvalues with
(b) the **unstable subspace** T^u is the direct sum of the generalized

eigenspaces corresponding to all the eigenvalues with

(c) the **centre subspace** T^c is the direct sum of the generalized eigenspaces corresponding to all eigenvalues with

Of course, all three subspaces T^s, T^u, T^c are invariant subspaces. The direct sum decomposition of the state space into invariant subspaces

gives much of the qualitative (i.e. topological) behaviour we are usually interested in, such as stability.

Asymptotic behaviour refers to qualitative behaviour of solutions $x(t) = \varphi(t, 0, x_0)$ as $t \to +\infty$ or $t \to -\infty$. The linear flow e^{At} (or the matrix A, or the linear vector field Ax, or the autonomous ODE $\dot{x} = Ax$) is called **hyperbolic** if $T^c = \{0\}$, i.e. if $\mathbb{R}^n = T^s \oplus T^u$, i.e. if A has no eigenvalue with zero real part, i.e. $\text{Re}(\lambda_j) \neq 0$ for all eigenvalues of A. In the hyperbolic case, asymptotic behaviour is easy to determine (e.g. Homework Assignment 1).

Notions of stablility: Let $x(t) = \varphi(t, 0, x_0)$, where $x(0) = x_0$. An invariant set S_0 is

(a) **Lyapunov stable** if for any (e.g. sufficiently small) open set U containing S_0, there exists an open set V containing S_0 such that $x_0 \in V$

implies $x(t) \in U$ for all $t \in [0, +\infty)$;

(b) **asymptotically stable** (or **attracting**) if there exists an open set U_0 containing S_0 such that $x_0 \in U_0$ implies $\text{dist}(x(t), S_0) \to 0$ as $t \to +\infty$;

(c) **stable** if both (a) and (b) are true;

(d) **unstable** if (a) is false.

In (b), $\text{dist}(x(t), S_0) = \inf\{\, \|x(t) - y\| : \; y \in S_0 \,\}$. Note that, according to these definitions (from the textbook), "unstable" is *NOT the same as* "not stable" (an invariant set could be both not stable and not unstable). The textbook (p. 17) gives an example (nonlinear "SNIC") of an invariant set that is asymptotically stable but is not Lyapunov stable. More common are examples (see Homework Assignment 1) that are Lyapunov stable but are not asymptotically stable.

The singleton set consisting of an **equilibrium** (a solution of $Ax = 0$) is an important example of an invariant set.

If a linear vector field is hyperbolic (and therefore, since zero is not an eigenvalue, the equilibrium 0 is unique), then the stability of the equilibrium 0 can be determined directly from the signs of real parts of the eigenvalues, and details of the real normal form of A in this case are not so important for asymptotic behaviour or stability.

i. If $T^s = \mathbb{R}^n$ (all eigenvalues have strictly negative real parts) then the equilibrium 0 is

ii. If $T^u \neq \{0\}$ (some eigenvalue has a strictly positive real part) then the equilibrium 0 is

If $\dot{x} = Ax$ is nonhyperbolic and $x(0) = x_0 \in T^c$ then the qualitative (topological) behaviour of $x(t) = \varphi(t, 0, x_0)$ can depend on some details of the real normal form of A.

Example 1.G. Topological classification of all 2-dimensional linear flows, by trace and determinant of the coefficient matrix. See also Example 2.3 (p. 48) in the textbook. Consider

$$\dot{x} = Ax, \quad x \in \mathbb{R}^2,$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is a real 2×2 matrix.

The characteristic polynomial is easily verified to be

$$h(\lambda) = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = \lambda^2 - \sigma\lambda + \Delta,$$

where

$\sigma =$ is the trace of A,

$\Delta =$ is the determinant of A.

(If you do not already know that $\mathrm{tr}(A) = \lambda_1 + \lambda_2$ and $\det(A) = \lambda_1\lambda_2$ where λ_1, λ_2 are the eigenvalues of A, it is worth verifying for yourself.)

We start with the nonhyperbolic case.

Case [iv]: <u>nonhyperbolic</u> ($\mathrm{Re}(\lambda_j) = 0$ for at least one eigenvalue)

There are four subcases

(a) purely imaginary eigenvalues

($b_\pm$) simple zero eigenvalue

(c) double zero eigenvalue

Plot [iv] in the (σ, Δ)-plane:

The three open regions in the (σ, Δ)-plane complementary to [iv] correspond to the hyperbolic cases.

Case [i]: hyperbolic sink $\sigma < 0$, $\Delta > 0$ ($\text{Re}(\lambda_j) < 0$ for both eigenvalues). The origin 0 is the unique equilibrium, and it is stable.

("stable nodes", "stable foci", "stable improper nodes" all have the *same* topological classification as hyperbolic sinks.)

Case [ii]: hyperbolic source $\sigma > 0$, $\Delta > 0$ ($\text{Re}(\lambda_j) > 0$ for both eigenvalues). 0 is an unstable equilibrium.

Case [iii]: hyperbolic saddle $\Delta < 0$ ($\lambda_1 < 0 < \lambda_2$). 0 is an unstable equilibrium.

For the hyperbolic cases [i], [ii], [iii] the particular details of the real normal form do not affect the topological classification.

But, for all the nonhyperbolic subcases of [iv], we need to examine the real normal form in more detail:

$[\mathrm{iv}]_a$ (purely imaginary eigenvalues, "linear centre"): the real normal form and its exponential are

$$R = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}, \quad e^{Rt} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix},$$

where $\omega = \sqrt{\Delta} > 0$. The phase portrait in normal form coordinates, and a typical phase portrait in original coordinates:

The origin 0 is the unique equilibrium, it is Lyapunov stable but not stable (because it is not asymptotically stable).

$[\mathrm{iv}]_{b\pm}$ (simple zero eigenvalue): the real normal form and its exponential are

$$R = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}, \quad e^{Rt} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}.$$

Phase portrait in normal form coordinates:

There is a *line* of equilibria. If $\sigma < 0$, each equilibrium is Lyapunov stable but not stable. If $\sigma > 0$, each equilibrium is unstable. (What is the stability of the invariant set – a line – consisting of *all* the equilibria?)

$[\text{iv}]_c$ (double zero eigenvalue): there are two further (sub-)subcases depending on the real normal form

$[\text{iv}]_{c(g)}$ ("generic"):

$$R = \left(\qquad\qquad \right), \quad e^{Rt} = \left(\qquad\qquad \right).$$

with the phase portrait in normal form coordinates:

There is a *line* of equilibria. Each equilibrium is unstable.

[iv]$_{c(n)}$ ("nongeneric"):

$$R = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}, \quad e^{Rt} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}.$$

There is a *plane* of equilibria. Each equilibrium is Lyapunov stable but not stable.

Completed classification diagram in the (σ, Δ)-plane