MATH 552 (2023W1) Lecture 5: Fri Sep 15

[**Last lecture:** Invariant sets and invariant subspaces, stable/unstable/centre subspaces for linear flows, hyperbolicity, various notions of stability, equilibria, topological classification of 2-dimensional linear flows (Example 1.G)

$$\dot{x} = Ax, \quad x \in \mathbb{R}^2,$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is a real 2×2 matrix.

The characteristic polynomial is

$$h(\lambda) = \lambda^2 - \sigma\lambda + \Delta,$$

$$\sigma = \mathrm{tr}(A) = a_{11} + a_{22} \quad \text{is the trace of } A,$$

$$\Delta = \det(A) = a_{11}a_{22} - a_{12}a_{21} \quad \text{is the determinant of } A.$$

(it is useful to recall $\mathrm{tr}(A) = \lambda_1 + \lambda_2$, $\det(A) = \lambda_1\lambda_2$)

Case [iv]: <u>nonhyperbolic</u> ($\mathrm{Re}(\lambda_j) = 0$ for at least one eigenvalue)

Four subcases

(a) purely imaginary eigenvalues $\sigma = 0$, $\Delta > 0$ ($\lambda_{1,2} = \pm i\sqrt{\Delta}$)

(b$_\pm$) simple zero eigenvalue $\Delta = 0$, $\sigma > 0$ or < 0 ($\lambda_1 = 0$, $\lambda_2 = \sigma \neq 0$)

(c) double zero eigenvalue $\Delta = 0$, $\sigma = 0$ ($\lambda_1 = 0$, $\lambda_2 = 0$)

Plot [iv] in the (σ, Δ)-plane, the three open regions in the (σ, Δ)-plane complementary to [iv] correspond to the hyperbolic cases.

Case [i]: <u>hyperbolic sink</u> $\sigma < 0$, $\Delta > 0$ ($\mathrm{Re}(\lambda_j) < 0$ for both eigenvalues). The origin 0 is the unique equilibrium, and it is stable.

("stable nodes", "stable foci", "stable improper nodes" all have the *same* topological classification as hyperbolic sinks.)

Case [ii]: <u>hyperbolic source</u> $\sigma > 0$, $\Delta > 0$ ($\mathrm{Re}(\lambda_j) > 0$ for both eigenvalues). 0 is an unstable equilibrium.

("unstable nodes", "unstable foci", "unstable improper nodes" all have the *same* topological classification as hyperbolic sources.)

Case [iii]: <u>hyperbolic saddle</u> $\Delta < 0$ ($\lambda_1 < 0 < \lambda_2$). 0 is an unstable equilibrium.

For the hyperbolic cases [i], [ii], [iii] the particular details of the real normal form do not affect the topological classification.]

But, for all the nonhyperbolic subcases of [iv], we need to examine the real normal form in more detail:

[iv]$_a$ (purely imaginary eigenvalues, "linear centre"): the real normal form and its exponential are

$$R = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}, \quad e^{Rt} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix},$$

where $\omega = \sqrt{\Delta} > 0$. The phase portrait in normal form coordinates, and a typical phase portrait in original coordinates:

The origin 0 is the unique equilibrium, it is Lyapunov stable but not stable (because it is not asymptotically stable).

[iv]$_{b\pm}$ (simple zero eigenvalue): the real normal form and its exponential are

$$R = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}, \quad e^{Rt} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}.$$

Phase portrait in normal form coordinates:

There is a *line* of equilibria. If $\sigma < 0$, each equilibrium is Lyapunov stable but not stable. If $\sigma > 0$, each equilibrium is unstable. (What is the stability of the invariant set – a line – consisting of *all* the equilibria?)

$[\text{iv}]_c$ (double zero eigenvalue): there are two further (sub-)subcases depending on the real normal form

$[\text{iv}]_{c(g)}$ ("generic"):

$$R = \left(\qquad\qquad \right), \quad e^{Rt} = \left(\qquad\qquad \right).$$

with the phase portrait in normal form coordinates:

There is a *line* of equilibria. Each equilibrium is unstable.

$[\mathrm{iv}]_{c(n)}$ ("nongeneric"):

$$R = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}, \quad e^{Rt} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}.$$

There is a *plane* of equilibria. Each equilibrium is Lyapunov stable but not stable.

Completed classification diagram in the (σ, Δ)-plane

Linear maps, linear discrete-time dynamical systems

A constant real $n \times n$ matrix A gives a **linear map**

$$x \mapsto A\,x, \quad x \in \mathbb{R}^n \tag{1.8}$$

(often $A\,x$, or the matrix A itself, is called the linear map; or sometimes the matrix A is said to **generate** the linear map (1.8)). The linear map is also a **linear diffeomorphism** if $\det(A) \neq 0$. Given any initial value $x_0 \in \mathbb{R}^n$ we generate a sequence of values in $\mathbb{R}^n$ with the **linear recursion**

$$x_{k+1} = A\,x_k, \quad k \in \mathbb{Z}.$$

For example (assuming $\det(A) \neq 0$)

It is easy to show that $x_k = A^k x_0$, where if $k = 0$ then $A^0 = I_n$, if $k > 0$ then A^k are positive integer powers of the matrix A, or if $k < 0$ then A^k are positive integer powers of the inverse matrix A^{-1} for a linear diffeomorphism. Assuming $\det(A) \neq 0$, the powers A^k of A, with $k \in \mathbb{Z}$, are **linear evolution operators**, and the triple $\{\mathbb{Z}, \mathbb{R}^n, A^k\}$ is a linear **discrete-time dynamical system** ($\mathbb{Z}$ is the **time set**, $\mathbb{R}^n$ is the **state space**, $\{A^k\}_{k\in\mathbb{Z}}$ is the family of linear evolution operators).

For brevity we will refer to a linear discrete-time dynamical system or a linear recursion as a “linear map” and assume implicitly that $\det(A) \neq 0$, unless explicitly noted otherwise, or sometimes as a “linear diffeomorphism” if we want to remind the reader explicitly that $\det(A) \neq 0$.

The **orbit** of a point $x_0 \in \mathbb{R}^n$ for a linear map is

a discrete set of points. (If $\det(A) = 0$, we can define a ”positive semi-orbit” for nonnegative k.)

It is possible to find integer powers A^k explicitly, by finding a basis of generalized eigenvectors of A and using this basis to determine a linear nonsingular coordinate change that takes the matrix A into its real normal form; the integer powers (*both* positive *and* negative) for a matrix in real normal form can be computed explicitly by hand (see Homework Assignment 1).

Invariant sets, hyperbolicity and stability for linear maps

A set $S \subseteq \mathbb{R}^n$ is (a) an **invariant** set, (b) a **positively invariant** set, etc. if $x_0 \in S$ implies $x_k = A^k x_0 \in S$ for all (a) $k \in \mathbb{Z}$, (b) $k \in \mathbb{Z} \cap [0, +\infty)$, etc.

For a linear map, the eigenvalues μ_j of the generating matrix A are often called the **multipliers** of A. Qualitative properties of a linear map are determined in "most" cases by the moduli (absolute values) $|\mu_j|$ of the multipliers μ_j.

Stable, unstable and centre subspaces: For any linear diffeomorphism $x \mapsto A\,x$ (i.e. $x_{k+1} = A\,x_k$)

(a) the **stable subspace** T^s is the direct sum of the generalized eigenspaces corresponding to all the multipliers with

(b) the **unstable subspace** T^u is the direct sum of the generalized eigenspaces corresponding to all the multipliers with

(c) the **centre subspace** T^c is the direct sum of the generalized eigenspaces corresponding to all the multipliers with

The direct sum decomposition into invariant subspaces

$$\mathbb{R}^n = T^s \oplus T^u \oplus T^c$$

gives much of the qualitative information we are usually interested in. The linear map $x \mapsto A\,x$ is called **hyperbolic** if $T^c = \{0\}$, i.e. if there is no multiplier on the unit circle in the complex plane, i.e. $|\mu_j| \neq 1$ for all multipliers of the linear map.

Notions of stability: Stability definitions for discrete-time systems are similar to those for continuous-time systems (**Exercise:** write them out).

The singleton set consisting of a **fixed point** (a solution of $A\,x = x$, or equivalently, of $(A - I_n)\,x = 0$) is an invariant set.

If a linear map is hyperbolic (then since 1 is not a multiplier, the fixed point 0 is unique), then the stability of the fixed point can be determined directly from the multipliers, and details of the real normal form in this case are not important for asymptotic behaviour, or stability.

i. If $T^s = \mathbb{R}^n$ (all multipliers are strictly inside the unit circle) then the fixed point 0 is stable.

ii. If $T^u \neq \{0\}$ (some multiplier is strictly outside the unit circle) then the fixed point 0 is unstable.

If $x \mapsto A\,x$ is nonhyperbolic and $x_0 \in T^c$ then the qualitative (topological) behaviour of $x_k = A^k\,x_0$ may depend on some details of the real normal form of A (e.g. see Homework Assignment 1).