MATH 552 (2023W1) Lecture 6: Mon Sep 18

[**Last lecture:** ...topological classification of 2-dimensional linear flows (Example 1.G); Linear discrete-time dynamical systems – basic concepts.]

Example 1.H. Topological classification of all 1-dimensional linear diffeomorphisms. See also Example 2.4 (p. 50) in the textbook. Consider

$$x \mapsto \mu\, x, \quad x \in \mathbb{R}^1 \quad (\mu \neq 0).$$

Fixed points: solve

$$\mu\, x = x,$$

i.e.

$$(\mu - 1)\, x = 0,$$

thus, if $\mu \neq 1$ then 0 is the unique fixed point, while if $\mu = 1$ then every point x (or x_0) is fixed.

Dynamics: if we think of μ as a 1×1 matrix, it has eigenvalue μ, so μ is the multiplier of the 1-dimensional linear map. Consider the recursion $x_{k+1} = \mu\, x_k$:

It is easy to see that

$$x_k = \mu^k x_0, \quad k \in \mathbb{Z}.$$

Case [iii]: nonhyperbolic ($|\mu| = 1$)

There are two subcases

(+)

(−)

Plot [iii] in the "punctured" μ-axis, $\mathbb{R}\backslash\{0\}$ ($\mu = 0$ is punctured out of the diagram):

There are four open intervals in $\mathbb{R}\backslash\{0\}$ that are complementary to the set corresponding to [iii], and these open intervals correspond to the hyperbolic cases.

Case $[\mathrm{i}]_+$: hyperbolic orientation-preserving sink $0 < \mu < 1$.

Case $[\mathrm{i}]_-$: hyperbolic orientation-reversing sink $-1 < \mu < 0$.

Case $[\mathrm{ii}]_+$: hyperbolic orientation-preserving source $1 < \mu$.

Case $[\mathrm{ii}]_-$: hyperbolic orientation-reversing source $\mu < -1$.

In cases $[\mathrm{i}]_\pm$ we have $|\mu| < 1$, 0 is a stable fixed point.

In cases $[\mathrm{ii}]_\pm$ we have $|\mu| > 1$, 0 is an unstable fixed point.

Completed classification diagram in the punctured μ-axis

Dynamics (phase portraits) can be visualized using **staircase/cobweb** diagrams:

Dynamics of the nonhyperbolic cases are easy to determine directly.

$[\text{iii}]_+$ $\mu = 1$ $(x \mapsto x)$: every point x is fixed, every point is Liapunov stable but not stable.

$[\text{iii}]_-$ $\mu = -1$ $(x \mapsto -x)$: the origin 0 is the unique fixed point, it is Liapunov stable but not stable.

In this last case $[\text{iii}]_-$ consider the **second iterate** map

$$x_{k+2} = x_k, \quad \text{i.e.} \quad x \mapsto x \quad \text{every 2nd iterate}$$

For the second iterate map, every point is fixed, thus every point belongs to a 2-**cycle**, an orbit with period 2

$$\{x_0, -x_0\}$$

If $x_0 \neq 0$, then this 2-cycle is **nontrivial** (i.e. consists of 2 distinct points).

Exercise.* Make a topological classification of all 2-dimensional linear *diffeomorphisms.* Draw a diagram in the "cut" (σ, Δ)-plane showing the classification (σ is the trace, $\Delta \neq 0$ is the determinant of the coefficient matrix, the σ-axis $\Delta = 0$ is cut out of the diagram).

Floquet multipliers for homogeneous linear periodic ODEs

If a homogeneous linear system of ODEs is *periodic*, we can make a useful correspondence with a discrete-time dynamical system, where the corresponding discrete-time unit is one period of the ODE system.

Consider a homogeneous linear, periodic, system of ODEs

$$\dot{x} = A(t)\, x, \quad x \in \mathbb{R}^n, \tag{1.9}$$

where $A(t)$ is a real continuous periodic $n \times n$ matrix with period $T_0 > 0$,

$$A(t + T_0) = A(t) \quad \text{for all } t \in \mathbb{R}.$$

To determine the qualitative properties of the linear periodic ODE (1.9), and in particular the stability of the solution $x(t) \equiv 0$ (i.e. the constant vector function defined by $x(t) = 0 \in \mathbb{R}^n$ for all $t \in \mathbb{R}$), we choose an initial time 0 (any initial time works, 0 is convenient), find the principal matrix $M(t, 0)$ at initial time 0, and define the **monodromy matrix** (at initial time 0) to be

$$M(T_0) = M(T_0, 0),$$

i.e. the monodromy matrix is the principal matrix evaluated at one period after the initial time.

Recall that the solution of the initial value problem for (1.9) with initial condition $x(0) = x_0$ is $x(t) = \varphi(t, 0, x_0) = M(t, 0)\, x_0$.

After one period, $x(T_0) = M(T_0, 0)\, x_0 = M(T_0)\, x_0$.

After two periods

Exercise. Show that $x(kT_0) = (M(T_0))^k\, x_0$, $k \in \mathbb{Z}$.

Using the fact that $x(t)$ is continuous in every closed interval $[\, kT_0, (k+1)T_0\,]$, it can be shown that the stability of the solution $x(t) \equiv 0$ for (1.9) is the same as the stability of the fixed point 0 for the linear map $x \mapsto M(T_0)\, x$.

It can be proved that the eigenvalues μ_j of the monodromy matrix $M(T_0)$ *do not depend* on which initial time ($t_0 = 0$, or some other t_0) is used in the definition. These eigenvalues are called the **Floquet mul-**

tipliers of (1.9), and a **Floquet exponent** is a number λ_j such that $e^{\lambda_j} = \mu_j$, where μ_j is a Floquet multiplier.

In the special case that $A(t)$ is actually a constant matrix, then (1.9) is actually autonomous, and then the eigenvalues of the constant matrix A are Floquet exponents.

Warning: if $A(t)$ is **not** constant, *the eigenvalues* $\lambda_j(t)$ *of* $A(t)$ *do* **not**, *in general, predict stability correctly* and furthermore, it can be difficult to find the principal matrix $M(t, 0)$ explicitly.

Exercise. Find the eigenvalues $\lambda_{1,2}(t)$ of $A(t)$, and the Floquet multipliers and Floquet exponents for Example 1.A. (Notice that the eigenvalues are not Floquet exponents, and the eigenvalues do not predict stability correctly.)

If $A(t)$ depends continuously on a parameter, then so do the Floquet multipliers. This fact can sometimes be used, to determine qualitative properties of a periodic system (1.9), by using simple perturbation arguments (perturbing the parameter).

2. Nonlinear Dynamical Systems