MATH 552 (2023W1) Lecture 7: Fri Sep 22

[**Last lecture:** ... linear discrete-time dynamical systems; Floquet multipliers for continuous-time periodic systems.]

2. Nonlinear Dynamical Systems

Existence, uniqueness and smooth dependence

First we review some ODE theory. Let $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ have a domain that is an open subset $U \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$, and consider a (paremetrized) *family* of initial value problems for systems of ODEs

$$\dot{x} = f(t, x, \alpha), \quad x(t_0) = x_0. \tag{2.1}$$

A fundamental theorem on ODEs is

Theorem 2.1. *If f is C^p ($p \geq 1$), and $(t_0, x_0, \alpha) \in U$, then each member of the family of initial value problems (2.1) has a unique solution $x(t) = \varphi(t, t_0, x_0, \alpha)$ defined for t belonging to some unique* *maximal open interval of existence* $\mathcal{J} = \mathcal{J}(t_0, x_0, \alpha) \subseteq \mathbb{R}$ *that contains* t_0*, and in general depends on* (t_0, x_0, α)*. Moreover, the solution* $\varphi(t, t_0, x_0, \alpha)$ *is* C^p *in all its variables* (t, t_0, x_0, α)*.*

We ignore parameter dependence for the rest of this chapter. Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ have a domain that is an open subset $U \subseteq \mathbb{R} \times \mathbb{R}^n$, and consider the initial value problem for a single system of ODEs

$$\dot{x} = f(t, x), \quad x(t_0) = x_0. \tag{2.2}$$

We note that Theorem 2.1 can be applied to the existence, uniqueness and smooth dependence of the solution $x(t) = \varphi(t, t_0, x_0)$ of (2.2). For $(t_0, x_0) \in U$, the unique maximally defined **solution curve** containing (t_0, x_0) is

Autonomous systems of ODEs and flows

If $f : \mathbb{R}^n \to \mathbb{R}^n$, with a domain that is an open subset $U \subseteq \mathbb{R}^n$, does not depend explicitly on t, then the system of ODEs $\dot{x} = f(x)$ is called **autonomous**. In this autonomous case, *without loss of generality* (see Homework Assignment 2) we can assume the initial time is zero,

$$t_0 = 0,$$

and for autonomous systems we consider the initial value problems

$$\dot{x} = f(x), \quad x(0) = x_0, \tag{2.3}$$

where x_0 belongs to the domain of f. In (2.3) we call f the **vector field**, we call the x-values **states**, and the domain of f is the **state space**. The collection of the unique maximally defined solutions $\varphi(t, 0, x_0)$ for all possible initial value problems (2.3) is called the **local flow** (**generated by** the vector field f) **in** $\mathbb{R}^n$. For autonomous systems we use the notation

$$\varphi^t(x_0) = \varphi(t, 0, x_0)$$

to denote the solutions of initial value problems (2.3), and we call each φ^t an **evolution operator**.

If the vector field f is C^p on its domain, then for each fixed t, the evolution operator φ^t is a *local* C^p *diffeomorphism in* $\mathbb{R}^n$, with a domain that is an open subset of the domain of f. The collection of evolution operators satisfy

$$\varphi^0 = \mathrm{id}, \tag{DS.0}$$

$$\varphi^{s+t} = \varphi^s \circ \varphi^t \quad \text{for all } s, t \in \mathbb{R}, \tag{DS.1}$$

whenever both sides are defined, where id is the identity map in $\mathbb{R}^n$, i.e. $\mathrm{id}(x) = x$ for all $x \in \mathbb{R}^n$.

Example 2.A. $\dot{x} = -x^2$, $x(0) = x_0 \in \mathbb{R}^1$.

By elementary methods, if $x_0 > 0$, the unique maximally defined solution of the initial value problem is

Exercise. Find explicity $\varphi^t(x_0)$ and $\mathcal{J}(0, x_0)$, if $x_0 < 0$, or if $x_0 = 0$.

If $X = \mathbb{R}^n$ (or more generally, if X is a manifold) and if $\varphi^t(x_0)$ is defined for *all* $x_0 \in X$ and *all* $t \in \mathbb{R}$ (i.e. $\mathcal{J}(0, x_0) = \mathbb{R}$ for all $x_0 \in X$) then the local flow in X is called a **global flow on** X, properties (DS.0)–(DS.1) hold *everywhere* on X without restrictions, and the triple $\{\mathbb{R}, X, \varphi^t\}$ is called a **continuous-time dynamical system**. For brevity we will usually refer to a local flow or a global flow or a continuous-time dynamical system simply as a "flow", and denote it by φ^t or $\dot{x} = f(x)$. Sometimes, we will consider flows on manifolds X other than $\mathbb{R}^n$ (examples later).

For a flow, the **orbit** of a state x_0 is

oriented by the direction of increasing t, and the **phase portrait** of a flow is the partitioning of the state space into orbits. Orbits $Or(x_0)$ are projections of solution curves $Cr(0, x_0)$ onto the state space (these projections are well defined for autonomous ODEs/flows).

A subset $S \subseteq X$ is

(a) an **invariant** set,

(b) a **positively** invariant set,

(c) a **negatively** invariant set, or

(c) a **locally** invariant set,

if $x(0) = x_0 \in S$ implies $x(t) = \varphi^t(x_0) \in S$ (i.e. $\varphi^t(S) \subseteq S$) for all

(a)

(b)

(c)

(d)

respectively.

An invariant set S is

(a) **Lyapunov stable** if for any open subset $U \subseteq X$ containing S, there exists an open subset $V \subseteq X$ containing S, such that $x(0) = x_0 \in V$ implies $x(t) = \varphi^t(x_0) \in U$ for all $t \in [0, +\infty)$ (notice we must have $[0, +\infty) \subseteq \mathcal{J}(0, x_0)$ for all $x_0 \in V$ for this definition);

(b) **asymptotically stable** if there exists an open subset $U \subseteq X$ containing S such that $x(0) = x_0 \in U$ implies $\operatorname{dist}(\varphi^t(x_0), S) \to 0$ as $t \to +\infty$;

(c) **stable** if both (a) and (b) are true;

(d) **unstable** if (a) is false.

An **equilibrium** is a solution $x = p^0$, of

$$f(x) = 0,$$

and the singleton set $S = \{p^0\}$, consisting of an equilibrium p^0, is clearly an invariant set.

Maps

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^p $(p \geq 1)$ local diffeomorphism in $\mathbb{R}^n$ with a domain that is an open subset of $\mathbb{R}^n$, and consider the map

$$x \mapsto f(x), \tag{2.4}$$

or equivalently the **recursion**

$$x_{k+1} = f(x_k), \quad k \in \mathbb{Z}. \tag{2.4'}$$

so we find that

$$x_k = f^k(x_0),$$

where $f^0 = \text{id}$ is the identity map, $f^1 = f$, $f^2 = f \circ f$, $f^3 = f \circ f \circ f$, etc.; and f^{-1} is the inverse map, $f^{-2} = f^{-1} \circ f^{-1}$, $f^{-3} = f^{-1} \circ f^{-1} \circ f^{-1}$, etc. Each k-th iterate map f^k, $k \in \mathbb{Z}$, is called an **evolution operator**. The evolution operators satisfy

$$f^0 = \text{id}, \tag{DS.0}$$

$$f^{j+k} = f^j \circ f^k \quad \text{for all } j, k \in \mathbb{Z}, \tag{DS.1}$$

whenever both sides are defined. A global diffeomorphism f on a manifold X generates a **discrete-time dynamical system** $\{\mathbb{Z}, X, f^k\}$ where the family of **evolution operators** $\{f^k\}_{k\in\mathbb{Z}}$ are the iterates of f (or of its inverse f^{-1}).

If f is a local diffeomorphism but not a global diffeomorphism, then $f^k(x_0)$ might not be defined for all x_0 or for all $k \in \mathbb{Z}$. If f is not invertible, we can still consider f^k for nonegative integers k. For brevity we will often call a local diffeomorphism or a global diffeopmorphism or a discrete-time dynamical system, simply a "map" or "diffeomorphism", and denote it by $x \mapsto f(x)$ or $x_{k+1} = f(x_k)$. We can consider maps in manifolds X other than $\mathbb{R}^n$.

Orbits, **phase portraits**, **invariant** sets for maps have similar definitions as for flows. (**Exercise.**)

A **fixed point** for a map is a solution $x = p^0$, of

and the set $\{p^0\}$ consisting of a fixed point p^0 is an invariant set.

The definitions for **Lyapunov stable**, **asymptotically stable**, **stable** and **unstable** invariant sets for maps are similar to the corresponding definitions for flows. (**Exercise.**)

Linearization and hyperbolicity

(a) *Linearization and hyperbolicity for flows:* Suppose p^0 is an equilibrium of

$$\dot{x} = f(x).$$

Near p^0, make a simple change of coordinates

Under this coordinate change, $\dot{x} = f(x)$ is transformed into

Suppose we ignore the higher order terms $o(\|\xi\|)$ (later we will see if this is advisable, or not!): the **linearization** (or **variational equation**) **at** p^0 is the linear vector field

$$\dot{\xi} = A\,\xi, \quad \text{where } A = f_x(p^0). \tag{2.5}$$

An equilibrium p^0 is called **hyperbolic** if the linearization (2.5) is hyperbolic, i.e. if Re $\lambda_j \neq 0$ for all eigenvalues λ_j of the constant real $n \times n$ matrix $f_x(p^0)$. We often call the eigenvalues of the matrix $f_x(p^0)$ the "eigenvalues of the equilibrium" p^0.