MATH 552 (2023W1) Lecture 8: Mon Sep 25

[**Last lecture:** ODE theory. Autonomous ODEs and flows. Maps.]

Linearization and hyperbolicity

(a) *Linearization and hyperbolicity for flows:* Suppose p^0 is an equilibrium of

$$\dot{x} = f(x).$$

Near p^0, make a simple change of coordinates

Under this coordinate change, $\dot{x} = f(x)$ is transformed into

Suppose we ignore the higher order terms $o(\|\xi\|)$ (later we will see if this is advisable, or not!): the **linearization** (or **variational equation**) **at** p^0 is the linear vector field

$$\dot{\xi} = A\,\xi, \quad \text{where } A = f_x(p^0). \tag{2.5}$$

An equilibrium p^0 is called **hyperbolic** if the linearization (2.5) is hyperbolic, i.e. if Re $\lambda_j \neq 0$ for all eigenvalues λ_j of the constant real $n \times n$

matrix $f_x(p^0)$. We often call the eigenvalues of the matrix $f_x(p^0)$ the "eigenvalues of the equilibrium" p^0.

(b) *Linearization and hyperbolicity for maps:* Suppose p^0 is a fixed point of

$$x \mapsto f(x).$$

Near p^0, make the coordinate change

then $x_{k+1} = f(x_k)$ is transformed into

Suppose we ignore the higher order terms $o(\|\xi\|)$: the **linearization** of $x \mapsto f(x)$ **at** p^0 is the linear map

$$\xi \mapsto A\,\xi, \quad \text{where } A = f_x(p^0). \tag{2.6}$$

A fixed point p^0 is called **hyperbolic** if the linearization (2.6) is hyperbolic, i.e. $|\mu_j| \neq 1$ for all multipliers (eigenvalues) μ_j of the constant real $n \times n$ matrix $f_x(p^0)$. For maps, the multipliers of $A = f_x(p^0)$ are called the **multipliers** of the linearization (2.6), and we ofen call them the "multipliers of the fixed point" p^0.

Topological equivalence

For two different dynamical systems (both continuous-time, or both discrete-time), we define more precisely what we mean when we say that they have "qualitatively the same dynamics". A dynamical system is **topologically equivalent** to another dynamical system if there exists a homeomorphism mapping the orbits of the first dynamical system onto the orbits of the second dynamical system, preserving the orientation of time.

Topological equivalence may be **local** or **global**, depending on whether the homeomorphism is local or global. In this course, mostly we are concerned with local topological equivalence.

For maps, topological equivalence can be shown to be equivalent to topological conjugacy: two maps f and g are **topologically conjugate** if there is a homeomorphism h such that $h \circ f = g \circ h$. Topological conjugacy may be **local** or **global**.

If an equilibrium or a fixed point is *hyperbolic*, then the linearization indeed gives a reliable qualitative description of the local dynamics. In fact, at a hyperbolic equilibrium or fixed point, in turns out that one only needs to know a few things about the eigenvalues of the linearization, in order to determine whether two systems are locally topologically equivalent.

For a *flow* near a hyperbolic equilibrium, local topological equivalence is determined by counting signs of the real parts of the eigenvalues (including multiplicities) of the linearization:

Theorem 2.2. *Two flows at hyperbolic equilibria are locally topologically equivalent if and only if the linearizations at their respective equilibria have the same dimensions* $n_- = \dim T^s$ *and* $n_+ = \dim T^u$*, of stable and unstable subspaces.*

For a map near a hyperbolic fixed point, local topological equivalence is slightly more complicated:

Theorem 2.3. *Two maps at hyperbolic fixed points are locally topologically equivalent if and only if their linearizations at their respective fixed points have (a) the same dimensions* $n_- = \dim T^s$ *and* $n_+ = \dim T^u$*, of stable and unstable subspaces; and (b) the same signs of the products, of all multipliers with* $|\mu_j| < 1$ *and of all multipliers with* $|\mu_j| > 1$*.*

Theorems 2.2 and 2.3 imply the **Hartman-Grobman Theorem**: a dynamical system at a *hyperbolic* equilibrium or fixed point is locally topologically equivalent to its linearization at the origin ($\xi = 0$), and the Hartman-Grobman Theorem implies the **Principle of Linearized Stability**: the stability of a *hyperbolic* equilibrium or fixed point is the same as the stability of (the origin for) its linearization.

Notice that, for continuous-time dynamical systems, topological equivalence is defined in terms of the flows. In applications, it is usually easier to compare vector fields directly, rather than their flows (a vector field is often given explicitly as ODEs, but the flow is the collection of *all the solutions* of the ODEs). Unfortunately, there is no easy general way to decide if two flows are topologically equivalent by studying only their vector fields (but, two important exceptions are i. linear flows, and ii. nonlinear flows near hyperbolic equilibria). However, there are some sufficient conditions for topological equivalence that are useful in practice:

(a) Suppose we have $\dot{y} = g(y)$ and smoothly *change spatial variables* $y = h(x)$ to get (using the chain rule, etc.) the corresponding, equivalent $\dot{x} = f(x)$. How are the vector fields f and g related?

This relationship between the vector fields is called smooth equivalence: if f and g are two vector fields and there is some C^p $(p \geq 1)$ diffeomorphism h such that

$$f(x) = M(x)^{-1} g(h(x)) \quad \text{for all } x, \quad \text{where } M(x) = h_x(x),$$

then we say f is **smoothly equivalent** (or C^p **equivalent** or C^p **diffeomorphic**) to g. Smooth equivalence implies topological equivalence.

(b) Suppose $\frac{dx(t)}{dt} = f(x(t))$ and we smoothly *rescale time* $x(s) = x(t)$ with $\frac{ds}{dt} = \mu(x(t)) > 0$, to get the equivalent $\frac{dx(s)}{ds} = g(x(s))$, then

We call this relationship between the vector fields orbital equivalence: if f and g are two vector fields, we say f is **orbitally equivalent** to g if there is some smooth, real-valued, *strictly positive* function $\mu : \mathbb{R}^n \to (0, \infty)$ such that

$$f(x) = \mu(x)\, g(x) \quad \text{for all } x.$$

Exercise. Show that orbital equivalence implies topological equivalence.

Of course, (a) and (b) can be combined.

Linearization at cycles

(a) *Linearization of flows, at cycles:* Let T_0 be a positive real number, and suppose

$$\dot{x} = f(x) \tag{*}$$

has a **cycle** (or **periodic orbit**) of period T_0 (or a T_0**-cycle**)

$$L_0 = \{p^0(t)\}_{t \in \mathbb{R}}$$

consisting of a solution $p^0(t)$ such that

To study the dynamics near this cycle we make a "moving" change of coordinates

$$x = p^0(t) + u, \quad (\|u\| \text{ small})$$

and then the vector field (*) is transformed:

Suppose we neglect the higher order terms $o(\|u\|)$. The **variational equation** (or **linearization**) of the vector field at $p^0(t)$ (or at L_0) is

$$\dot{u} = A(t)\, u, \quad \text{where } A(t) = f_x(p^0(t)). \tag{2.7}$$

Since $A(t) = f_x(p^0(t))$ is a continuous periodic real $n \times n$ matrix with period T_0, we consider the Floquet multipliers.

If L_0 is a T_0-cycle for a flow (*), then (**Exercise**) *one* of the Floquet multipliers of the linearization at $p^0(t)$ (2.7) *must* be equal to 1 , so the n Floquet multipliers, counting multiplicities, are

$$1\,,\ \mu_1\,,\ \ldots\ ,\ \mu_{n-1}$$

(not necessarily all distinct). The $n-1$ **nontrivial** Floquet multipliers $\mu_1, \ldots, \mu_{n-1}$ contain linearized stability information for the cycle. The cycle L_0 is called **hyperbolic** if none of the nontrivial Floquet multipliers is on the unit circle $|\mu| = 1$.

A cycle L_0 for a flow is a **limit cycle** if there is an open set in the state space that contains L_0 but no other cycles. (If a cycle is hyperbolic, then it must be a limit cycle.)