MATH 552 (2023W1) Lecture 9: Wed Sep 27

[**Last lecture:** Linearization and hyperbolicity, topological equivalence, topological conjugacy, usefulness of linearization at hyperbolic equilibria or fixed points, smooth equivalence, orbital equivalence, linearization at cycles for flows]

Example 2.B. Let $\omega > 0$ be fixed: linearization at a hyperbolic cycle for

$$\dot{x}_1 = f_1(x_1, x_2) = x_1 - \omega x_2 - x_1^3 - x_1 x_2^2,$$
$$\dot{x}_2 = f_2(x_1, x_2) = \omega x_1 + x_2 - x_1^2 x_2 - x_2^3.$$

Exercise. Show that under the polar coordinates change of variables

$$x_1 = r\,\cos(\theta), \quad x_2 = r\,\sin(\theta)$$

the vector field (for $x \neq 0$) is transformed into the smoothly equivalent

where $\mathbb{R}_+ = (0, \infty)$. Solution:

$$\theta(t) = \qquad\qquad\qquad\qquad \pmod{2\pi}$$

and $r(t)$ could be found explicitly (**Exercise**), but here it is sufficient to plot r vs. $\dot{r}$ and thus find the one-dimensional phase portrait on the r-axis:

In $\mathbb{R}_+ \times \mathbb{S}^1$ we have a cycle

which in rectangular coordinates is

Next, we linearize at this cycle. First we compute the derivative

$$f_x(x) = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix},$$

then evaluate the derivative at the cycle

$$f_x(p^0(t)) = A(t) = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}.$$

Now we need to find the principal matrix for $\dot{u} = A(t)\,u$.

Exercise. Use the fact that $\dot{p}^0(t) = f(p^0(t))$ and the chain rule to show that $u(t) = \dot{p}^0(t)$ is, in general, always a solution of (2.7).

In this case

and we let

so we have

We need another (linearly independent) solution $\psi^{[1]}(t)$ of the variational equation such that $\psi^{[1]}(0) = (\ 1 \ \ 0\ )^{\mathsf{T}}$. A clue: linearizing $\dot{r} = r - r^3$ at $r^0(t) \equiv 1$ gives $\dot{\xi} = -2\xi$ which has solutions $\xi(t) = \text{constant} \ \cdot \ e^{-2t}$, so we try undetermined coefficients

and (eventually!) get a suitable solution

So the principal matrix (at $t_0 = 0$) is

$$M(t,0) = \left(\qquad\qquad\qquad\qquad \right).$$

Evaluating after one period $T_0 = 2\pi/\omega$, the monodromy matrix is

$$M(T_0) = M(2\pi/\omega, 0) = \left(\qquad\qquad\qquad \right),$$

and its eigenvalues are the Floquet multipliers

As must always be the case when linearizing at a cycle for an autonomous ODE, one of the Floquet multipliers is equal to 1. The remaining, non-trivial, Floquet multiplier is

and therefore the cycle is hyperbolic (and furthermore, it is stable).

(b) *Linearization of maps, at cycles:* Let K_0 be a positive integer, and suppose the map

$$x \mapsto f(x)$$

has a **cycle** of **period** K_0, or a K_0**-cycle**: a set of discrete points

such that

Note that each point p_j^0 in the cycle L_0 is a fixed point of the K_0**th iterate** map

$$x \mapsto f^{K_0}(x)$$

where

To study the linearized stability of a K_0-cycle L_0 for $x \mapsto f(x)$, we simply linearize the K_0th iterate map $x \mapsto f^{K_0}(x)$ at any one of its K_0 fixed points p_j^0. We say the cycle L_0 is **hyperbolic** if one of the points in the K_0-cycle for the map $x \mapsto f(x)$ is a hyperbolic fixed point for the K_0th iterate map $x \mapsto f^{K_0}(x)$.

Exercise. Prove that for this definition it does not matter which point in the cycle is chosen.

The process of finding linearizations, determining eigenvalues or multipliers, and noting the implications (or lack of them!) for nonlinear systems, is often referred to as "linearized stablility analysis".

Poincaré maps

Suppose the flow generated by

$$\dot{x} = f(x)$$

has a (nonconstant) cycle $L_0 = \{p^0(t)\}$, with *least* period $T_0 > 0$. Near L_0, we can study the fully nonlinear (but sometimes only local) continuous-time dynamics by means of a discrete-time **Poincaré map**, in a space of one less dimension.

First, choose any point $p_0^0 = p^0(0)$ on the cycle. Then $p^0(t) \neq p_0^0$ for all $t \in (0, T_0)$, and $p^0(T_0) = p_0^0 = p^0(0)$.

Next, choose a **cross-section Σ at p_0^0**: a smooth $(n-1)$-dimensional manifold Σ that contains p_0^0, such that

$$f(p_0^0) \notin T_{p_0^0}\Sigma \quad \text{(i.e. the vector } f(p_0^0) \text{ is not tangent to } \Sigma \text{ at } p_0^0\text{)}$$

A convenient choice for Σ often has the form

$$\Sigma = \{\, x \in \mathbb{R}^n : \ g(x) = 0 \,\},$$

where $g : \mathbb{R}^n \to \mathbb{R}^1$ is some smooth scalar function with

$$g(p_0^0) = 0, \quad \langle \nabla g(p_0^0), f(p_0^0) \rangle \neq 0.$$

Then, choose an inital value x_0 in Σ, and solve the initial value problem for $\dot{x} = f(x)$, $x(0) = x_0$, to generate the unique maximally defined solution $x(t) = \varphi^t(x_0)$. By continuity of solutions with respect to the initial value (Theorem 2.1), if x_0 is sufficiently near p_0^0, then $x(t) = \varphi^t(x_0)$ will *first* return to Σ at some instant of time $T(x_0) > 0$, the "first return time", near the period T_0 of the cycle (in fact, $T(p_0^0) = T_0$).

Define the Poincaré map $P : \Sigma \to \Sigma$ by

Therefore, p_0^0 is a fixed point of P. If the vector field f is C^p, then the Poincaré map P is a local C^p diffeomorphism, with a domain that is an open set U *in* Σ that contains p_0^0 (i.e. U is *relatively open* in Σ).