MATH 552 (2023W1) Lecture 10: Fri Sep 29

[**Last lecture:** ... linearization at cycles for flows and maps, Poincaré maps ...]

Vector field

$$\dot{x} = f(x)$$

Cycle $L_0 = \{p^0(t)\}$, with *least* period $T_0 > 0$.

Choose a point $p_0^0 = p^0(0)$ on the cycle.

Choose a **cross-section** Σ **at** p_0^0.

Now, choose an inital value x_0 in Σ, and solve the initial value problem for $\dot{x} = f(x)$, $x(0) = x_0$, to generate the unique maximally defined solution $x(t) = \varphi^t(x_0)$. By continuity of solutions with respect to the initial value (Theorem 2.1), if x_0 is sufficiently near p_0^0, then $x(t) = \varphi^t(x_0)$ will *first* return to Σ at some instant of time $T(x_0) > 0$, the "first return time", near the period T_0 of the cycle (in fact, $T(p_0^0) = T_0$).

Define the Poincaré map $P : \Sigma \to \Sigma$ by

Therefore, p_0^0 is a fixed point of P. If the vector field f is C^p, then the Poincaré map P is a local C^p diffeomorphism, with a domain that is an open set U *in* Σ that contains p_0^0 (i.e. U is *relatively open* in Σ).

To make explicit calculations, it is often convenient to define some smooth local coordinates $\xi = (\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$ on Σ.

Arranging for $\xi = 0 \in \mathbb{R}^{n-1}$ to correspond to $x = p_0^0 \in \Sigma$, we can express the Poincaré map in coordinate form as

now with a fixed point $0 \in \mathbb{R}^{n-1}$. Notice we are using the same symbol P to denote both the Poincaré map as we originally defined it, and the expression of that map in a particular coordinate system.

The stability of a cycle $\{p^0(t)\}$ for the flow corresponds to the stability of the fixed point for the Poincaré map. Linearized stability of the fixed point is determined by the multipliers (eigenvalues) $\mu_1, \ldots, \mu_{n-1}$ of the linearization of the Poincaré map at its fixed point $\xi = 0$, the $(n-1) \times (n-1)$ matrix $P_\xi(0)$. These multipliers can be shown to be independent of the choice of the point p_0^0 on the cycle, of the choice of the cross-section Σ at p_0^0, and of the choice of the coordinates ξ on Σ. In fact, the following can be proved:

Theorem 2.4. *The nontrivial Floquet multipliers of the continuous-time linearization* $\dot{u} = f_x(p^0(t))\, u$*, of the flow of* $\dot{x} = f(x)$ *at the cycle* $\{p^0(t)\}_{t\in\mathbb{R}}$ *in* $\mathbb{R}^n$*, are the same as the multipliers of the linearization* $\eta \mapsto P_\xi(0)\, \eta$*, of the Poincaré map* $\xi \mapsto P(\xi)$ *at the corresponding fixed point* 0 *in* $\mathbb{R}^{n-1}$*.*

Thus, a cycle for a flow is hyperbolic if and only if the corresponding fixed point for a Poincaré map is hyperbolic.

Example 2.C. (See Example 2.B.)

$$\dot{x}_1 = x_1 - \omega x_2 - x_1^3 - x_1 x_2^2,$$

$$\dot{x}_2 = \omega x_1 + x_2 - x_1^2 x_2 - x_2^3,$$

where $\omega > 0$ is fixed.

Recall in Example 2.B we found a cycle for this system

$$p^0(t) = (x_1^0(t), x_2^0(t)) = (\cos(\omega t), \sin(\omega t)).$$

Now we construct a Poincaré map for this cycle. Choose a point on the cycle

and a cross-section

Exercise. Verify that Σ is indeed a cross-section.

We define a coordinate $\xi_1 \in \mathbb{R}^1$ for Σ, by

so that $\xi_1 = 0$ corresponds to the intersection of the cycle and the cross-section. Next, we have to solve the initial value problem with initial value parametrized by ξ_1. This is most easily done in polar coordinates: solve

The explicit solution (**Exercise**) is

which in rectangular coordinates is

In polar coordinates, the cross-section Σ is represented by

so the time of first return is the time it takes for $\theta(t)$ to go from 0 to 2π, which is (independent of x_0, due to the simplicity of this example). Then the Poincaré map in rectangular coordinates is

which is easily seen to belong to Σ. The coordinate representation of this Poincaré map is

We could plot a staircase diagram and phase portrait for P

recalling that $\xi_1 = 0$ corresponds to the point where the cycle intersects the cross-section.

For a linearized stability analysis, we can explicitly compute the derivative (**Exercise**) of the Poincaré map P (in the coordinate representation) and we can verify we get the same value as the nontrivial Floquet multiplier in Example 2.B, as guaranteed by Theorem 2.4.

Example 2.D. (The Poincaré map, the implicit function theorem, and continuity of eigenvalues can establish the existence and linearization of a "perturbed" periodic solution.)

Consider a forced, damped nonlinear oscillator

$$\ddot{u} + \delta\,\dot{u} - u + u^3 = \varepsilon\,\cos(\omega t), \tag{2.D.1}$$

where δ, ω are given (fixed) positive parameters and ε is a small perturbation parameter.

The standard conversion to a 2-dimensional first-order (nonautonomous) periodic system is obtained by letting $x_1 = u$, $x_2 = \dot{u}$ to get

and then we convert to a 3-dimensional first-order *autonomous* system with the simple trick of letting $\theta = \omega t \pmod{2\pi}$ in $\mathbb{S}^1$:

and the phase space $X = \mathbb{R}^2 \times \mathbb{S}^1$ is a 3-dimensional manifold.

When $\varepsilon = 0$, observe that the original, second-order nonautonomous ODE (2.D.1) has three constant (and therefore trivially periodic) solutions $u^0(t, 0) \equiv 0, +1, -1$. In the 3-dimensional representation (2.D.3), these correspond to three cycles

But, when $\varepsilon \neq 0$, *none* of these are solutions (**Exercise:** verify!) Using a Poincaré map, we can show that all three periodic solutions $u^0(t, 0)$, that exist for $\varepsilon = 0$, "persist" as $O(|\varepsilon|)$-close periodic solutions $u^0(t, \varepsilon)$ for all $\varepsilon \neq 0$ with $|\varepsilon|$ sufficiently small.

To show this persistence, define a "global" cross-section

Let us focus on one of the cycles when $\varepsilon = 0$,

(the analysis for the other two cycles is very similar). For any initial condition in Σ we solve (at least in principle, the solution exists) the initial value problem for (2.D.3) with initial condition