MATH 552 (2023W1) Lecture 11: Wed Oct 4

[**Last lecture:** ...Poincaré maps...]

Example 2.D, continued. Recall the nonautonomous 2nd-order ODE

$$\ddot{u} + \delta\,\dot{u} - u + u^3 = \varepsilon\,\cos(\omega t), \tag{2.D.1}$$

or the equivalent, autonomous 3-dimensional system of 1st-order ODEs

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - \delta x_2 - x_1^3 + \varepsilon\,\cos(\theta) \\ \dot{\theta} &= \omega \end{aligned} \tag{2.D.3}$$

for $\tilde{x} = (x_1, x_2, \theta)$ belonging to the 3-dimensional manifold $X = \mathbb{R}^2 \times \mathbb{S}^1$.

When $\varepsilon = 0$, (2.D.1) has three constant (and therefore trivially periodic) solutions $u^0(t, 0) \equiv 0, +1, -1$.

"Global" cross-section

$$\Sigma = \{\tilde{x} = (x_1, x_2, \theta) \in X :\ (x_1, x_2) \in \mathbb{R}^2,\ \theta = 0\,(\mathrm{mod}\ 2\pi)\}$$

One of the three cycles when $\varepsilon = 0$ is

$$p_0^0(t, 0) = (0, 0, \omega t\,(\mathrm{mod}\ 2\pi))$$

For any initial condition in Σ

$$\tilde{x}_0 = (\xi_1, \xi_2, 0\,(\mathrm{mod}\ 2\pi))$$

we solve (at least in principle, the solution exists) the initial value problem for (2.D.3) and there exists a solution of the form

The time of first return is constant

so

and a coordinate representation of the Poincaré map is

where $(\xi_1, \xi_2) = (0, 0) \in \mathbb{R}^2$ corresponds to the point $(0, 0, 0 \pmod{2\pi}) \in X$ where, when $\varepsilon = 0$, the cycle intersects the cross-section Σ.

For $\varepsilon = 0$, we already know $(\xi_1, \xi_2) = (0, 0)$ is a fixed point of the Poincaré map

We will show that this fixed point "persists", at least for small $\varepsilon \neq 0$, i.e. P has a *unique* fixed point, when $\varepsilon \neq 0$, that is $O(|\varepsilon|)$-close to the fixed point that exists when $\varepsilon = 0$. To show this, we use a standard perturbation argument using the (multivariable) *Implicit Function Theorem* (Appendix A).

Let $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathbb{R}^2$, define $F : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2$, by

so that to find a fixed point of $P(\cdot, \epsilon)$ we solve, equivalently,

Then F is smooth near $(0,0) \in \mathbb{R}^2 \times \mathbb{R}^1$,

so to apply the implicit function theorem (Appendix A) we need to check that the 2×2 matrix $F_\xi(0,0)$ is nonsingular, i.e.

i.e. the 2×2 matrix $P_\xi(0,0)$ does not have a multiplier $\mu_j(0) = 1$.

We find the multipliers of $P_\xi(0,0)$, by computing the Floquet multipliers of the variational equation about the cycle. Set $\varepsilon = 0$ and linearize the autonomous system (2.D.3) in X about the cycle $(0, 0, \omega t \pmod{2\pi})$
Exercise: let $x_1 = 0 + u_1$, $x_2 = 0 + u_2$, $\theta = \omega t + u_3$ and linearize in (u_1, u_2, u_3)) to get the variational equation

or

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

Conveniently, when $\varepsilon = 0$ this variational equation is actually *autonomous* so in this special case the principal matrix is

and the monodromy matrix at $t_0 = 0$ is

so for $\varepsilon = 0$ the Floquet multipliers (i.e. the eigenvalues of $M(T_0, 0)$) are

where λ_1, λ_2 are the eigenvalues of

$$\begin{pmatrix} & \\ & \end{pmatrix}$$

This 2×2 matrix has determinant Δ so we immediately know

and therefore the nontrivial Floquet multipliers (which are the same as the multipliers/eigenvalues of $P_\xi(0,0)$) satisfy

and thus $F_\xi(0,0)$ is indeed nonsingular.

Now, we have checked that the implicit function theorem applies, and we conclude that $F(\xi,\varepsilon) = 0$ has a unique locally defined smooth solution, a unique map $\xi^0 : \mathbb{R}^1 \to \mathbb{R}^2$ defined and smooth for all $\varepsilon \in \mathbb{R}^1$ sufficiently close to 0 with $\xi^0(0) = 0 \in \mathbb{R}^2$, such that $F(\xi^0(\varepsilon),\varepsilon) = 0$, i.e. $\xi^0(\varepsilon)$ is a fixed point for the Poincaré map $P(\cdot,\varepsilon)$,

$$P(\xi^0(\varepsilon),\varepsilon) = \xi^0(\varepsilon) \quad \text{for all } \varepsilon \text{ sufficiently close to } 0$$

Since $\xi^0(\varepsilon)$ is smooth and $\xi^0(0) = 0$, we know that $\xi^0(\varepsilon) = O(|\varepsilon|)$. (This justifies a formal expansion $\xi^0(\varepsilon) = \xi^0_1\,\varepsilon + \xi^0_2\,\varepsilon^2 + \cdots$ using only positive integer powers of ε.)

We can also use a continuity argument to find the multipliers of the linearization, at least accurately enough to determine the topological type of the fixed point. Since the 2×2 matrix $P_\xi(\xi^0(\varepsilon), \varepsilon)$ depends continuously on ε near $\varepsilon = 0$, it can be proved that the multipliers $\mu_j(\varepsilon)$, $j = 1, 2$ of the matrix depend continuously on ε near 0 (in fact the dependence of the multipliers is smooth, because the dependence of the matrix is smooth *and the multipliers are distinct*; but in general if it is possible that the multipliers could be multiple and the real normal form could be not a diagonal matrix, the dependence of the multipliers can only be assumed to be continuous). Only assuming continuous dependence of multipliers on ε is enough to conclude that

and for $\varepsilon \neq 0$ (ε near 0) the fixed point has the topological type of an orientation-preserving (unstable) saddle, the same as for $\varepsilon = 0$. Thus the corresponding cycle $L_\varepsilon = \{\, p^0(t, \varepsilon) = (x_1^0(t, \varepsilon), x_2^0(t, \varepsilon), \omega t \,(\mathrm{mod}\ 2\pi)) \,\}_{t \in \mathbb{R}}$ for (2.D.3), with initial condition $p^0(0, \varepsilon) = (\xi^0(\varepsilon), 0 \,(\mathrm{mod}\ 2\pi))$, has the corresponding topological type (it is hyperbolic and unstable).

We can conclude that for all ε sufficiently close to 0, the original forced oscillator (2.D.1) has a unique $(2\pi/\omega)$-periodic solution $u^0(t,\varepsilon)$ that is $O(|\varepsilon|)$-close to the unperturbed solution $u^0(t,0) \equiv 0$, depending smoothly on ε, so we could expand in a power series in ε to find approximations. Moreover, this periodic solution is unstable, of saddle type.

Exercise: Perform a similar analysis for the periodic solution $u^0(t,0) \equiv 1$, under the perturbation $\varepsilon \neq 0$ (it also persists as a $(2\pi/\omega)$-periodic solution $u^0(t,\varepsilon) = 1 + O(|\varepsilon|)$, but it is stable, of sink type). Of course $u^0(t,0) \equiv -1$ can be analyzed in the same way.

Arguments of this type (using the implicit function theorem and continuity of eigenvalues or multipliers on parameters) are standard to show that *hyperbolic* equilibria, fixed points, or cycles are locally "structurally stable": for all sufficiently small perturbations to the system, the perturbed system is locally topologically equivalent to the unperturbed system. For more complicated invariant sets, if they are hyperbolic by some appropriate definition, local structural stability can be shown using essentially the same arguments.