MATH 552 (2023W1) Lecture 12: Fri Oct 6

[**Last lecture:** ...Poincaré maps...]

Example 2.D, continued.

$$\ddot{u} + \delta\,\dot{u} - u + u^3 = \varepsilon\,\cos(\omega t), \tag{2.D.1}$$

or, equivalently,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - \delta x_2 - x_1^3 + \varepsilon\,\cos(\theta) \\ \dot{\theta} &= \omega \end{aligned} \tag{2.D.3}$$

Poincaré map

$$P : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2, \quad P(\xi_1, \xi_1, \varepsilon) = \begin{pmatrix} x_1(2\pi/\omega, \xi_1, \xi_2, \varepsilon) \\ x_2(2\pi/\omega, \xi_1, \xi_2, \varepsilon) \end{pmatrix}$$

has, for $\varepsilon = 0$, a fixed point at the origin in $\mathbb{R}^2$:

$$P(0,0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For $\varepsilon \neq 0$, close to 0, we showed, by carefully using the implicit function theorem, that $P(\,\cdot\,, \varepsilon)$ still has a fixed point in $\mathbb{R}^2$, and it is $O(|\varepsilon|)$-close to the origin. (For $\varepsilon = 0$, the variational equation is autonomous so it is much easier to find Floquet multipliers.)

We also used a continuity argument to find the multipliers of the linearization, at least accurately enough to determine the topological type of the fixed point: Since the 2×2 matrix $P_\xi(\xi^0(\varepsilon), \varepsilon)$ depends continuously on ε near $\varepsilon = 0$, it can be proved that the multipliers $\mu_j(\varepsilon)$, $j = 1, 2$ of the matrix depend at least continuously on ε near 0 (in fact, the dependence of the multipliers is smooth, because the dependence of the matrix is smooth *and the multipliers are distinct*). Only assuming continuous dependence of multipliers on ε is enough to conclude that

$$\mu_1(\varepsilon),\ \mu_2(\varepsilon) \text{ are both real}, \quad 0 < \mu_1(\varepsilon) < 1 < \mu_2(\varepsilon),$$

and for $\varepsilon \neq 0$ (ε sufficiently near 0) the fixed point has the topological type of an orientation-preserving (unstable) saddle, the same as for $\varepsilon = 0$. Thus the corresponding cycle $L_\varepsilon = \{\, p^0(t, \varepsilon) = (x_1^0(t, \varepsilon), x_2^0(t, \varepsilon), \omega t \pmod{2\pi}) \,\}_{t \in \mathbb{R}}$ for (2.D.3), with initial condition $p^0(0, \varepsilon) = (\xi^0(\varepsilon), 0 \pmod{2\pi})$, has the corresponding topological type (it is hyperbolic and unstable).

We therefore conclude that for all ε sufficiently close to 0, the original forced oscillator (2.D.1) has a unique $(2\pi/\omega)$-periodic solution $u^0(t,\varepsilon)$ that is $O(|\varepsilon|)$-close to the unperturbed solution $u^0(t,0) \equiv 0$, depending smoothly on ε, so we could expand in a power series in ε to find approximations. Moreover, this periodic solution is unstable, of saddle type.

Exercise: Perform a similar analysis for the periodic solution $u^0(t,0) \equiv 1$, under the perturbation $\varepsilon \neq 0$ (it also persists as a $(2\pi/\omega)$-periodic solution $u^0(t,\varepsilon) = 1 + O(|\varepsilon|)$, but it is stable, of sink type). Of course $u^0(t,0) \equiv -1$ can be analyzed in the same way.

Arguments of this type (using the implicit function theorem and continuity of eigenvalues or multipliers on parameters) are standard, to show that *hyperbolic* equilibria, fixed points, or cycles are locally "structurally stable": for all sufficiently small perturbations to the system, the perturbed system is locally topologically equivalent to the unperturbed system. For more complicated invariant sets, if they are hyperbolic by some appropriate definition, local structural stability can be shown using essentially the same arguments.

Introduction to invariant manifolds

If S is an invariant set for a flow or for a map, then its **stable set** $W^s(S)$ is the set of all states whose orbits approach S in forward time, e.g. for a flow

while its **unstable set** $W^u(S)$ is the set of all states whose orbits approach S in backward time. Both the stable set and the unstable set are themselves (**Exercise**) invariant sets.

Local invariant manifolds at hyperbolic equilibria or fixed points:

If $S = \{p^0\}$ and p^0 is a hyperbolic equilibrium, we have further results.

Theorem 2.5 (Local stable and unstable manifolds for flows)**.** *If f is C^p ($p \geq 1$) and p^0 is a hyperbolic equilibrium for $\dot{x} = f(x)$, then the intersections of $W^s(p^0)$ and $W^u(p^0)$ with a sufficiently small open neighbourhood of p^0 contain C^p submanifolds $W^s_{loc}(p^0)$ and $W^u_{loc}(p^0)$ of dimensions $n_- = \dim T^s$ and $n_+ = \dim T^u$, respectively.*

The smooth submanifolds $W^s_{loc}(p^0)$ and $W^u_{loc}(p^0)$ are tangent at p^0 to $T^s_{p^0}$ and $T^u_{p^0}$, the stable and unstable subspaces of the linearization at p^0, respectively.

Similarly, for hyperbolic fixed points:

Theorem 2.6 (Local stable and unstable manifolds for maps)**.** *If* f *is* C^p *(*$p \geq 1$*) and* p^0 *is a hyperbolic fixed point for* $x \mapsto f(x)$*, then the intersections of* $W^s(p^0)$ *and* $W^u(p^0)$ *with a sufficiently small open neighbourhood of* p^0 *contain* C^p *submanifolds* $W^s_{loc}(p^0)$ *and* $W^u_{loc}(p^0)$ *of dimensions* $n_- = \dim T^s$ *and* $n_+ = \dim T^u$*, respectively.*
The smooth submanifolds $W^s_{loc}(p^0)$ *and* $W^u_{loc}(p^0)$ *are tangent at* p^0 *to* $T^s_{p^0}$ *and* $T^u_{p^0}$*, the stable and unstable subspaces of the linearization at* p^0*, respectively.*

For families of dynamical systems (vector fields/flows, or maps) that depend smoothly (C^p for some $p \geq 1$) on parameters $\alpha \in \mathbb{R}^m$, hyperbolic equilibria, or hyperbolic fixed points, for all sufficiently small changes in α persist as families of hyperbolic equilibria, or hyperbolic fixed points, $p^0(\alpha)$ with the same topological type and the families depend smoothly on parameters (**Exercise:** how would you prove this? why hyperbolic?). Moreover, Theorems 2.5 and 2.6 can be extended, to prove that the corresponding families of local stable and unstable manifolds $W^s_{loc}(p^0(\alpha), \alpha)$ and $W^s_{loc}(p^0(\alpha), \alpha)$ also depend smoothly on parameters.

Example 2.E. (See Example 2.D.)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - \delta x_2 - x_1^3 + \varepsilon \cos(\theta) \\ \dot{\theta} &= \omega \end{aligned} \tag{2.D.3}$$

for $\tilde{x} = (x_1, x_2, \theta)$ belonging to the 3-dimensional manifold $X = \mathbb{R}^2 \times \mathbb{S}^1$.

In Example 2.D. we constructed a family of Poincaré maps $P(\,\cdot\,, \varepsilon) : \mathbb{R}^2 \to \mathbb{R}^2$ for (2.D.3), smoothly parametrized by $\varepsilon \in \mathbb{R}^1$, based on a global cross-section Σ. We proved that for some interval of ε-values about $\varepsilon = 0$, there is a family of fixed points $\xi^0(\varepsilon) \in \mathbb{R}^2$, smoothly parametrized by ε, and their topological types are all the same: orientation-preserving saddles ($0 < \mu_1(\varepsilon) < 1 < \mu_2(\varepsilon)$). So the linearizations (2×2 matrices) $P_\xi(\xi^0(\varepsilon), \varepsilon)$ all have 1-dimensional stable subspaces ($n_- = 1$) and 1-dimensional unstable subspaces ($n_+ = 1$). By Theorem 2.6, for each fixed ε (sufficiently close to 0), the nonlinear Poincaré map $P(\,\cdot\,, \varepsilon)$ has a 1-dimensional local stable manifold $W^s_{loc}(\xi^0(\varepsilon), \varepsilon)$ and a 1-dimensional local unstable manifold $W^u_{loc}(\xi^0(\varepsilon), \varepsilon)$. By the extended version of Theorem 2.6, if ε is allowed to vary, then the families $W^s_{loc}(\xi^0(\varepsilon), \varepsilon)$ and $W^u_{loc}(\xi^0(\varepsilon), \varepsilon)$ depend smoothly on the parameter ε, and therefore remain $O(|\varepsilon|)$-close to the "unperturbed" invariant manifolds for $\varepsilon = 0$.

Now, we use the points in $W^s_{loc}(\xi^0(\varepsilon), \varepsilon)$ and $W^u_{loc}(\xi^0(\varepsilon), \varepsilon)$ as initial values in Σ for (2.D.3) to generate local stable and unstable 2-dimensional manifolds $\tilde{W}^s_{loc}(L_\varepsilon, \varepsilon)$ and $\tilde{W}^u_{loc}(L_\varepsilon, \varepsilon)$ for the "perturbed" cycles L_ε. As ε varies (near 0), these 2-dimensional manifolds also depend smoothly on ε, and therefore remain $O(|\varepsilon|)$-close to the "unperturbed" invariant manifolds for $\varepsilon = 0$.

Global invariant manifolds:

If an equilibrium or fixed point p^0 is hyperbolic, then letting all states belonging to the positively invariant local stable manifold $W^s_{loc}(p^0)$ evolve backwards in time we recover the stable set $W^s(p^0)$, and similarly letting all states in the negatively invariant local unstable manifold $W^u_{loc}(p^0)$ evolve forwards in time we recover $W^u(p^0)$. This implies that $W^s(p^0)$ and $W^u(p^0)$ are not just invariant *sets*, but also are *locally C^p submanifolds*. For this reason (if p^0 is a *hyperbolic* equilibrium or fixed point) the stable and unstable sets $W^s(p^0)$ and $W^u(p^0)$ are usually referred to as the **(global) stable** and **unstable manifolds** of p^0, respectively. They are important features of the dynamics.