MATH 552 (2023W1) Lecture 13: Wed Oct 11

[**Last lecture:** ... Poincaré maps. Stable and unstable sets. Local stable and unstable manifolds (at hyperbolic equilibria or fixed points)]

Global invariant manifolds:

If an equilibrium or fixed point p^0 is hyperbolic, then letting all states belonging to the positively invariant local stable manifold $W^s_{loc}(p^0)$ evolve backwards in time we recover the stable set $W^s(p^0)$, and similarly letting all states in the negatively invariant local unstable manifold $W^u_{loc}(p^0)$ evolve forwards in time we recover $W^u(p^0)$. This implies that $W^s(p^0)$ and $W^u(p^0)$ are not just invariant *sets*, but also are *locally C^p submanifolds*. For this reason (if p^0 is a *hyperbolic* equilibrium or fixed point) the stable and unstable sets $W^s(p^0)$ and $W^u(p^0)$ are usually referred to as the **(global) stable** and **unstable manifolds** of p^0, respectively. They are important features of the dynamics.

Introduction to Hamiltonian systems

With additional structure in a vector field, it is often easier to determine global properties of the corresponding flow. We consider a vector field in the even-dimensional state space $\mathbb{R}^{2s}$, generated by a real-valued C^{r+1} ($r \geq 1$) **Hamiltonian (function)** H with domain in $\mathbb{R}^{2s}$. Let $q =$

$(q_1, \ldots, q_s) \in \mathbb{R}^s$, $p = (p_1, \ldots, p_s) \in \mathbb{R}^s$. The **Hamiltonian system**

$$\dot{x} = f(x), \quad x \in \mathbb{R}^{2s}$$

generated by the Hamiltonian $H : \mathbb{R}^{2s} \to \mathbb{R}$ has the form

$$\dot{q} = H_p(q, p), \quad \dot{p} = -H_q(q, p), \tag{2.8}$$

where $x = (q, p) \in \mathbb{R}^s \times \mathbb{R}^s = \mathbb{R}^{2s}$.

Exercise. Show that if $x(t) = (q(t), p(t))$ is a solution of (2.8), then $\frac{d}{dt} H(x(t)) \equiv 0$, i.e. $H(x(t)) \equiv$ constant.

So in a Hamiltonian system, *all solutions $x(t)$ remain on level sets of the Hamiltonian function $H(x)$* = constant. This property (often called "conservation of energy") makes determining the global phase portrait especially easy in the case of $s = 1$ (one "degree of freedom").

Example 2.F. An unforced undamped nonlinear oscillator

$$\ddot{u} - u + u^3 = 0. \tag{2.F.1}$$

Let $x_1 = u$ $(= q)$ ("position"), $x_2 = \dot{u}$ $(= p)$ ("momentum"), and write the equation as an equivalent 2-dimensional, autonomous system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_1 - x_1^3. \end{aligned} \tag{2.F.2}$$

We first find equilibria: $x_2 = 0$ and $x_1 - x_1^3 = 0$

Calculate the linearizations at each of the three equilibria and find (**Exercise**) that

By Theorem 2.2, $(0, 0)$ is in fact for the nonlinear system an unstable hyperbolic saddle and, by Theorem 2.5 and its consequences, it has stable and unstable manifolds. But Theorem 2.2 does not apply to $(\pm 1, 0)$ so we *cannot* yet jump to any conclusions about their stability and local dynamics, we need more analysis.

In some prerequisite course you might plot nullclines and the direction field (**Exercise**)

but, while useful, this is not yet enough to determine (up to local topological equivalence) the dynamics near $(\pm 1, 0)$.

Now we use the (observed) fact that the system is Hamiltonian, generated by

(check), so orbits of the sytem remain on the level sets

$$\tfrac{1}{2}x_2^2 - \tfrac{1}{2}x_1^2 + \tfrac{1}{4}x_1^4 = \text{constant}$$

To help see what these level sets look like, first plot the “potential energy” $y = H(x_1, 0)$

Then, observe that for any constant $x_1 = x_1^0$, the graph of $y = H(x_1^0, x_2) = \frac{1}{2}x_2^2 + \text{constant}$ is a parabola

These two plots (may) help to visualize the graph of the surface $y =$

$H(x_1, x_2)$ in 3 dimensions

and (the main goal) to plot the level curves

The orbits must remain on these level curves, and be oriented consistent with the direction field, so the global phase portrait is

The Hamiltonian analysis confirms that $(0,0)$ is a saddle, and now we see that $(\pm 1, 0)$ are at local minima of the Hamiltonian function H and therefore are locally topologically equivalent to centres for a linear system but occurring in a nonlinear system, i.e. "nonlinear centres". So they are Lyapunov stable but not asymptotically stable.

Caution: in general, purely imaginary eigenvalues for the linearization do *not* always imply a nonlinear centre! This example is a *Hamiltonian* system, which is both "nongeneric" and "structurally unstable".

We also note that $W^s((0,0)) \cap W^u((0,0)) \neq \{(0,0)\}$, there are two "homoclinic" orbits.

Example 2.G. The undamped planar pendulum.

Newton's 2nd law and rescaling time gives

$$\ddot{\phi} + \sin(\phi) = 0, \qquad \phi \in \mathbb{S}^1 \tag{2.G.1}$$

Let $\nu = \dot{\phi} \in \mathbb{R}^1$ and rewrite (2.G.1) as a vector field in a 2-dimensional manifold

$$\begin{aligned} \dot{\phi} &= \nu, \\ \dot{\nu} &= -\sin(\phi), \end{aligned} \tag{2.G.2}$$

where $x = (\phi, \nu)$ belongs to the *cylinder* $X = \mathbb{S}^1 \times \mathbb{R}^1$.

Equilibria: solve

and we find there are precisely *two* equilibria in X

Linearized stability (**Exercise**):

$p^0_{[1]}$ is

$p^0_{[2]}$ is

Nullclines and direction field (**Exercise**):

We observe that (2.G.2) is a Hamiltonian system:

$$\dot{\phi} = \nu = H_\nu, \quad \dot{\nu} = -\sin(\phi) = -H_\phi$$

where

$$H(\phi, \nu) = \tfrac{1}{2}\,\nu^2 - \cos(\phi)$$

so we should study the level sets $H(\phi, \nu) = \text{constant}$.