MATH 552 (2023W1) Lecture 14: Thu Oct 12

[**Last lecture:** ...Invariant manifolds. Hamiltonian systems...]

Example 2.G, continued, the undamped planar pendulum.

$$\begin{aligned} \dot{\phi} &= \nu, \\ \dot{\nu} &= -\sin(\phi), \end{aligned} \tag{2.G.2}$$

where $x = (\phi, \nu)$ belongs to the *cylinder* $X = \mathbb{S}^1 \times \mathbb{R}^1$.

Equilibria, linearized stability, nullclines, direction field.

Observe that (2.G.2) is a Hamiltonian system:

$$\dot{\phi} = \nu = H_\nu, \quad \dot{\nu} = -\sin(\phi) = -H_\phi$$

where

$$H(\phi, \nu) = \tfrac{1}{2}\,\nu^2 - \cos(\phi)$$

so we should study the level sets $H(\phi, \nu) = \text{constant}$.

For $\nu = 0$ we plot the “potential energy” function

$$y = H(\phi, 0) = -\cos(\phi)$$

For any constant $\phi = k$ we have

$$y = H(k, \nu) = \tfrac{1}{2}\,\nu^2 + \text{constant}$$

so we can visualize the graph in 3 dimensions

$$y = H(\phi, \nu)$$

which helps us to plot the level sets in 2 dimensions

$$H(\phi, \nu) = \text{constant}$$

Finally, combining with information from the direction field, we can plot the global phase portrait in the cylinder X

Locally, we know the equilibrium $p^0_{[1]}$ corresponds to a local minimum of H so it is a nonlinear centre, and therefore it is Liapunov stable but not asymptotically stable. Globally, we see there are continuous families of cycles which are not limit cycles. There are two types, which differ qualitatively (i.e. topologically) from each other: "librations" and "rotations". Moreover, separating these two types of cycles are two homoclinic orbits. The same phase portrait can be visualized in a 2-dimensional surface

$X \subseteq \mathbb{R}^3$

Lyapunov functions

A generalization of Hamiltonian functions, useful (when it works!) for both *local* and *global* analysis of flows. Consider a smooth vector field

$$\dot{x} = f(x), \quad x \in X$$

Suppose $L : X \to \mathbb{R}$ is a smooth function with a domain contained in X, and define

where $\langle \cdot\,,\, \cdot \rangle$ denotes the **inner product** (or dot product) of two vectors. If $x(t)$ is a solution, then notice that

If $\dot{L} \leq 0$ in some open set U, and, if for some real number c the closed set

$$\overline{R} = \{\, x \in X : \;\; L(x) \leq c \,\}$$

is contained in U, then we can say

A **Lyapunov function for** $\dot{x} = f(x)$ **on an open set** U in X is a smooth function $L : X \to \mathbb{R}$ such that

$$\dot{L} \leq 0 \quad \text{in } U.$$

For example, a Hamiltonian function is a Lyapunov function for the associated Hamiltonian vector field, on all of its domain.

A) *Locally*, Lyapunov functions can be useful to study stability of equilibria (esp. nonhyperbolic). Suppose p^0 is an equilibrium and L is a Lyapunov function with a domain that is some open set U that contains p^0.

If

i) $L(p^0)$ is an isolated local minimum of L, and

ii) $\dot{L} \leq 0$ in U,

then p^0 is

If, in addition to i) and ii) we have

iii) $\dot{L} < 0$ in $U \setminus \{p^0\}$,

then p^0 is

B) *Globally,* Lyapunov functions can be useful to study global aspects of flows. For example, a **trapping region** is a compact (in finite dimensions, same as closed and bounded) positively invariant subset with nonempty interior, and sometimes a trapping region can be found in the form

$$\overline{R} = \{\, x \in X : \;\; L(x) \leq c \,\}.$$

Example 2.H. (Compare Example 2.G.) The damped planar pendulum.

$$\ddot{\phi} + \delta\,\dot{\phi} + \sin(\phi) = 0, \quad \phi \in \mathbb{S}^1, \tag{2.H.1}$$

or equivalently

$$\left.\begin{aligned} \dot{\phi} &= \ \nu, \\ \dot{\nu} &= \ -\delta\,\nu - \sin(\phi), \end{aligned}\right\} \quad x = (\phi, \nu) \in \mathbb{S}^1 \times \mathbb{R}^1 = X \tag{2.H.2}$$

where $\delta > 0$ is a constant.

Equilibria (**Exercise**):

Linearized stability (**Exercise**):

so by Theorem 2.2, $p^0_{[1]}$ is stable and $p^0_{[2]}$ is unstable, and moreover we know their local topological types.

Nullclines and direction field (**Exercise**):

A global Lyapunov function is useful to analyze the global phase portrait of (2.H.2). Let L be the *same* Hamiltonian function from Example 2.G, i.e.

$$L(x) = L(\phi, \nu) = \tfrac{1}{2}\,\nu^2 - \cos(\phi), \quad x = (\phi, \nu) \in X.$$

We calculate

thus L is a Liapunov function on all of X (i.e. a global Liapunov function), and all closed sublevel sets of the form

$$\{\, x \in X : \;\; L(x) \le c \,\}$$

are positively invariant.

Exercise. Show that, if $x(t) = (\phi(t), \nu(t))$ is a *nonconstant* solution, then $g(t) = L(x(t))$ is a *strictly decreasing* function of t (i.e. $t_1 < t_2 \Rightarrow g(t_1) > g(t_2)$) (even if $\nu(t_0) = 0$ for some isolated t_0).

Then, by the exercise, all orbits that are not equilibria "move strictly downhill" on the global "contour map" of L.

It is helpful to locate the (global) stable and unstable manifolds $W^s(p^0_{[2]})$ and $W^u(p^0_{[2]})$, of the hyperbolic saddle equilibrium $p^0_{[2]}$, relative to the level sets of L. In 2 dimensions, the stable manifold of a hyperbolic saddle equilibrium sometimes forms a "separatrix" that separates regions in the flow that have different qualitative behaviour as $t \to \infty$.

Also, if $0 < \delta \ll 1$, then it is helpful to realize that by Theorem 2.1, for all *finite* t, the orbits of (2.H.2) stay $O(\delta)$-close to the orbits of the $\delta = 0$ system (2.G.2) which, recall, remain on the level sets of L.

Exercise. Determine and carefully sketch $W^s(p^0_{[1]})$, the (global) stable manifold of the hyperbolic sink equilibrium $p^0_{[1]}$.