MATH 552 (2022W1) Lecture 15: Fri Oct 13

[**Last lecture:** ... Hamiltonian functions. Lyapunov functions ...]

Example 2.H, continued. The damped planar pendulum.

$$\left.\begin{aligned} \dot{\phi} &= \nu, \\ \dot{\nu} &= -\delta\,\nu - \sin(\phi), \end{aligned}\right\} \quad x = (\phi, \nu) \in \mathbb{S}^1 \times \mathbb{R}^1 = X \qquad (2.H.2)$$

where $\delta > 0$ is a constant.

Equilibria, linearized stability, nullclines and direction field.

Global Lyapunov function

$$L(x) = L(\phi, \nu) = \tfrac{1}{2}\,\nu^2 - \cos(\phi), \quad x = (\phi, \nu) \in X,$$

with

$$\dot{L} \leq 0 \quad \text{in all of } X.$$

Exercise. Show that, if $x(t) = (\phi(t), \nu(t))$ is a *nonconstant* solution, then $g(t) = L(x(t))$ is a *strictly decreasing* function of t (i.e. $t_1 < t_2 \Rightarrow g(t_1) > g(t_2)$) (even if $\nu(t_0) = 0$ for some isolated t_0).

Then, by the exercise, all orbits that are not equilibria "move strictly downhill" on the global "contour map" of L.

It is helpful to locate the (global) stable and unstable manifolds $W^s(p^0_{[2]})$ and $W^u(p^0_{[2]})$, of the hyperbolic saddle equilibrium $p^0_{[2]}$, relative to the level sets of L. In 2 dimensions, the stable manifold of a hyperbolic saddle equilibrium sometimes forms a "separatrix" that separates regions in the flow that have different qualitative behaviour as $t \to \infty$.

Also, if $0 < \delta \ll 1$, then it is helpful to realize that by Theorem 2.1, for all *finite* t, the orbits of (2.H.2) stay $O(\delta)$-close to the orbits of the $\delta = 0$ system (2.G.2) which, recall, remain on the level sets of L.

Exercise. Determine and carefully sketch $W^s(p^0_{[1]})$, the (global) stable manifold of the hyperbolic sink equilibrium $p^0_{[1]}$.

Unlike Hamiltonian functions, Lyapunov functions need not be restricted to even-dimensional systems.

Example 2.I. Consider the (famous) *Lorenz system*

$$\dot{x_1} = \sigma\,(x_2 - x_1),$$
$$\dot{x_2} = r\,x_1 - x_2 - x_1 x_3, \tag{2.I.1}$$
$$\dot{x_3} = x_1 x_2 - \beta\, x_3,$$

for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, where σ, r, β are positive constants. Notice that the origin $0 = (0, 0, 0)$ is an equilibrium in $\mathbb{R}^3$.

Exercise. Show by linearized stability analysis that 0 is stable, if $0 < r < 1$ (i.e. locally stable).

We can use a Lyapunov function to show 0 is *globally* stable, if $0 < r < 1$. Define

for all $x \in \mathbb{R}^3$.

Exercise. Show that if $0 < r < 1$ and if $x \neq 0$, then $\dot{L}(x) < 0$.

Since the level sets of L are a continuous family of concentric ellipsoids about the origin $0 \in \mathbb{R}^3$, the result of this exercise implies that 0 is globally asymptotically stable, if $0 < r < 1$, i.e. *every* orbit approaches 0 as $t \to \infty$, and hence (**Exercise**) 0 is globally stable if $0 < r < 1$.

3. One-Parameter Local Bifurcations

Families of dynamical systems

In this chapter we study local topological changes in dynamical systems (vector fields/flows or maps), near equilibria or fixed points or cycles, that may occur if a system is perturbed a small amount. We can control the size of perturbations by varying parameters in a smoothly parametrized *family* of dynamical systems.

If an equilibrium, fixed point or cycle is *hyperbolic*, then under all sufficiently small perturbations of the system, such as a small change in parameter values, the system near the equilibrium, fixed point or cycle *does not change* up to local topological equivalence (see e.g. HW 3, problem 3). If a system does not change topologically under all sufficiently small perturbations, it is said to be *structurally stable* (or "persistent" or "robust").

So if we want to see topological *changes* under perturbations, we should consider systems that are not structurally stable. The simplest cases involve *nonhyperbolic* equilibria or fixed points.

Example 3.A. (See HW 2, problem 3.) $f : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$,

$$\dot{x} = f(x, \alpha) = \alpha\, x - x^2$$

$\alpha_0 = 0$ is a *bifurcation value*, there is a *local bifurcation* at $p_0^0 = 0$ (which is a nonhyperbolic equilibrium for $\alpha = \alpha_0$).

$$\dot{\alpha} = 0$$

$$\dot{x} = \alpha\, x - x^2$$

Phase portrait of 2-dimensional extended (α, x) *system*

Branching diagram (or bifurcation diagram)

Bifurcation diagram (or bifurcation set): a "parametric portrait" in parameter space together with corresponding phase portraits in state space.

Now consider m-parameter **families** of n-dimensional vector fields

$$\dot{x} = f(x, \alpha), \qquad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^m,$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is smooth. Here, the x-values are in n-dimensional state space $\mathbb{R}^n$ and the α-values are in m-dimensional parameter space $\mathbb{R}^m$. For each fixed α, we have a vector field (or autonomous ODE) in $\mathbb{R}^n$, and as α varies, the vector field changes smoothly. But the flow or phase portrait may change in some important way. A parameter value α_0 is a **bifurcation value** if, for every open neighbourhood of α_0 in $\mathbb{R}^m$, there is always some α_1 in that neighbourhood such that $\dot{x} = f(x, \alpha_0)$ and $\dot{x} = f(x, \alpha_1)$ are not topologically equivalent. A **bifurcation diagram** (or bifurcation set) is a **parametric portrait** (a stratification of the parameter space induced by topological equivalence in the state space), together with the corresponding phase portraits in state space. A **branching diagram** (or bifurcation diagram) is a diagram in parameter-state space $\mathbb{R}^m \times \mathbb{R}^n$ showing branches of equilibria $x = p^0_{[j]}(\alpha)$ (or cycles) and their stability (not so practical if $m > 1$ or $n > 1$). A **local** bifurcation is a bifurcation where the topological non-equivalence occurs in some (perhaps sufficently small) open neighbourhood of a point (typically an equilibrium) in state space $\mathbb{R}^n$. All these definitions can be adjusted if the state space is an n-dimensional manifold X, but for local bifurcations we may as well assume the state space is $\mathbb{R}^n$.

Similar definitions are made for smooth m-parameter families of n-dimensional maps

$$x \mapsto f(x, \alpha), \qquad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^m.$$

Topological equivalence of families

We compare smooth families of dynamical systems and define more precisely what we mean when we say families have "qualitatively the same" dynamics for corresponding parameter values.

Two families of vector fields

$$\frac{dx}{dt} = f(x, \alpha), \qquad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^m, \tag{3.1}$$

and

$$\frac{dy}{ds} = g(y, \beta), \qquad y \in \mathbb{R}^n, \quad \beta \in \mathbb{R}^m, \tag{3.2}$$

are defined to be **topologically equivalent** if there is a homeomorphism of parameter variables $p : \mathbb{R}^m \to \mathbb{R}^m$, $\beta = p(\alpha)$, and a family of homeomorphisms of state variables $h(\cdot, \alpha) : \mathbb{R}^n \to \mathbb{R}^n$, $y = h(x, \alpha)$, that map the orbits of (3.1) for parameter values α onto the orbits of (3.2) for parameter values $\beta = p(\alpha)$, preserving the orientation of time. The two families are **locally** topologically equivalent if p or $h(\cdot, \alpha)$ are local homeomorphisms, defined on open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively (typicially, on some sufficiently small open neighbourhoods of specific points).

Similarly, we can define topological equivalence and local topological equivalence for two families of maps,

$$x \mapsto f(x,\alpha), \qquad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^m,$$

and

$$y \mapsto g(y,\beta), \qquad y \in \mathbb{R}^n, \quad \beta \in \mathbb{R}^m.$$

From now on in this chapter, we focus on one-parameter families of systems, $m = 1$.