MATH 552 (2023W1) Lecture 16: Mon Oct 16

[**Last lecture:** ... Lyapunov functions. Families of dynamical systems. Topological equivalence of families ...]

Similarly, we can define topological equivalence and local topological equivalence for two families of maps,

$$x \mapsto f(x, \alpha), \qquad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^m,$$

and

$$y \mapsto g(y, \beta), \qquad y \in \mathbb{R}^n, \quad \beta \in \mathbb{R}^m.$$

From now on, we focus on one-parameter families of systems, $m = 1$.

The fold bifurcation for 1-dimensional vector fields

Suppose $f : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ is locally defined and smooth near a point $(p_0^0, \alpha_0) \in \mathbb{R}^1 \times \mathbb{R}^1$, and consider the one-parameter family of one-dimensional vector fields

$$\dot{x} = f(x, \alpha), \qquad x \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1. \tag{†}$$

For a local bifurcation, we look for an equilibrium p_0^0 that is nonhyperbolic for some specific parameter value α_0. In a one-dimensional vector field, there is only one way for an equilibrium to be nonhyperbolic: $\lambda_0 = 0$, where $\lambda_0 = f_x(p_0^0, \alpha_0)$ is the eigenvalue of the 1×1 linearization

$$\dot{\xi} = f_x(p_0^0, \alpha_0)\, \xi, \qquad \xi \in \mathbb{R}^1.$$

The two conditions

$$(\textit{equilibrium}), \tag{F.0.i}$$

$$(\textit{bifurcation}), \tag{F.0.ii}$$

give, for $\alpha = \alpha_0$, an equilbrium for (†) that: i) exists by (F.0.i), and ii) is nonhyperbolic by (F.0.ii). At this point, we only have a *suspected* bifurcation, since the phase portrait near a nonhyperbolic equilibrium is, typically, sensitive to arbitrarily small perturbations of the vector field $f(x, \alpha_0)$. We need to verify and specify this sensitivity.

For any fixed α, expanding $f(x, \alpha)$ in a one-variable Taylor series about p_0^0, we have

Then, letting α vary and expanding each α-dependent coefficient about α_0, and using (F.0.i) and (F.0.ii), we have and we see that $f_x(p_0^0, \alpha) = O(|\alpha - \alpha_0|)$.

For a **fold** bifurcation (also known as a **saddle-node** bifurcation), we require that two additional "generic" conditions hold:

$$(\textit{transversality}), \tag{F.1}$$

$$(\textit{nondegeneracy}). \tag{F.2}$$

[Given any family (†) that satisfies the equalities (F.0.i) and (F.0.ii), the inequalities (F.1) and (F.2) are satisfied "generically" – roughly speaking, if we choose an "arbitrary" family (†) that already satisfies (F.0.i) and (F.0.ii), then it is "highly probable" (in some sense that can be made mathematically rigourous, but is not worth taking the time to cover the required background) that it also satisfies (F.1) and (F.2).]

To simplify subsequent notation, assume (without loss of generality)

$$p_0^0 = 0, \quad \alpha_0 = 0.$$

Then, expanding $f(x, \alpha)$ in a two-variable Taylor series at $(p_0^0, \alpha_0) = (0, 0)$

and using the four conditions (F.0.i)–(F.2), the family (†) is

Then equilibria are obtained by solving $f(x, \alpha) = 0$ with the implicit function theorem to get a locally unique smooth curve of equilibria parametrized by x, for all x sufficiently near 0, as follows:

Equation for equilbria:

$$0 = f(x, \alpha) = a\,\alpha + b\,x^2 + O(|\alpha|^2 + |\alpha||x| + |x|^3),$$

where f is smooth. We have a known solution, by (F.0.i), to perturb from:

$$f(0, 0) = 0.$$

We have, by (F.1),

$$a = f_\alpha(0, 0) \neq 0$$

so we can use the implicit function theorem to solve for α in terms of x, a locally unique and smooth solution

$$\alpha = \alpha^0(x),$$

with

$$\alpha^0(0) = 0,$$

agreeing with the known solution. We have

$$f(x, \alpha^0(x)) = 0 \qquad \text{for all } x \text{ near } 0,$$

so we can expand in a Taylor series

$$\alpha^0(x) = \alpha_0^0 + \alpha_1^0\, x + \alpha_2^0\, x^2 + O(|x|^3),$$

substitute into the equation just above, and find the coefficients α_j^0 (**Exercise**)

$$\alpha = \alpha^0(x) = 0 + 0\, x + \left(-\frac{b}{a}\right)\, x^2 + O(|x|^3).$$

More generally we have for l x sufficiently near p_0^0,

$$\alpha = \alpha^0(x) = \alpha_0 - \frac{b}{a}\,(x - p_0^0)^{\,2} + O(|x - p_0^0|^{\,3}).$$

Notice that

$$\operatorname{sgn}(\alpha - \alpha_0) = \operatorname{sgn}\left(-\frac{b}{a}\right)$$

for $x \neq p_0^0$ and x near p_0^0.

If we are careful, we can now solve (**Exercise**)

for x in terms of α see that there are two branches of equilibria if $\alpha \neq \alpha_0$

Since f is continuous, in a sufficiently small open neighbourhood of (p_0^0, α_0), f can only change sign along the curve where $f = 0$, i.e. along the curve $\alpha = \alpha^0(x)$. In the complementary regions f must have a definite sign (e.g. for the case $a > 0$, $b < 0$):

Then we can draw the local phase portrait for the 2-dimensional system

$$\dot{\alpha} = 0$$

$$\dot{x} = f(x, \alpha)$$

(e.g. for the case $a > 0$, $b < 0$)

and the local branching diagram (e.g. for the case $a > 0$, $b < 0$)

and the local bifurcation diagram (for the case $a > 0$, $b < 0$)

To make local topological equivalence of families rigourous, we should construct local homeomorphisms p, $h(\cdot, \alpha)$ and prove that they do what is required (see the textbook p. 80 if you are interested). Here, we only state the theorem summarizing this work.

Theorem 3.1. *If $f : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ is C^3 in an open set containing a point (p_0^0, α^0) and satisfies the four conditions* (F.0.i)–(F.2)*, then the family of vector fields*

$$\frac{dx}{dt} = f(x, \alpha), \quad at \ (p_0^0, \alpha_0),$$

has a fold bifurcation, locally topologically equivalent to the normal form family of vector fields

$$\frac{dy}{ds} = a\,\beta + b\,y^2, \quad at \ (0, 0).$$

After the higher-order terms have been "transformed away" with suitable homeomorphisms (a continuously invertible change of parameters and a parametrized family of continuously invertible changes of state variables), the dynamics of the normal form are easily determined with explicit calculations.

Exercise. Draw the local branching and bifurcation diagrams for the other three cases of $a \neq 0$, $b \neq 0$.

By rescaling the state variable y, and rescaling and *possibly changing the sign* of the parameter β, the normal form above can be expressed (**Exercise**) even more simply as

$$\frac{d\eta}{ds} = \gamma \pm \eta^2.$$

The textbook calls this the **topological normal form** of the fold bifurcation.

Notice that in (†), the Taylor series terms of order

$$O(|\alpha - \alpha_0|^2 + |\alpha - \alpha_0||x - p_0^0| + |x - p_0^0|^3)$$

do not qualitatively affect the local dynamics, near (p_0^0, α_0). As long as $a \neq 0$ and $b \neq 0$, the original family (†) is locally topologically equivalent to the *normal form* family of vector fields which is obtained essentially by discarding (or "truncating") the specified higher-order terms in the Taylor series.

[**NOTE:** *Which* specific terms are considered "higher-order" and therefore can be "ignored" *depends* on the bifurcation being studied!]