MATH 552 (2023W1) Lecture 17: Wed Oct 18

[**Last lecture:** ... topological equivalence of families. Fold (saddle-node) bifurcation for flows/vector fields/ODEs.]

The symmetric pitchfork bifurcation for 1-dimensional vector fields

Many models have built-in symmetries, and these can change the types of bifurcations that are generic. A simple case of symmetry occurs when each member of a family of one-dimensional vector fields is an odd function of the state variable. Consider the family

$$\dot{x} = f(x, \alpha), \qquad x \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1.$$

If the family satisfies the symmetry condition that f is odd in x for all α,

$$(\mathit{symmetry}), \qquad \text{(SP.0.i)}$$

(one of two possible types of "$\mathbb{Z}_2$-equivariance"), then as a consequence of (SP.0.i) we have

$$f(0, \alpha) = 0 \quad \text{for all } \alpha \qquad (*)$$

i.e. $p^0_{[1]}(\alpha) \equiv 0$ is a branch of equilibria for all α. Then (*) implies that we can "factor out" x from $f(x, \alpha)$

and write

$$f(x,\alpha) = x\,\tilde{f}(x,\alpha),$$

where $\tilde{f}$ is smooth (enough) at $(0,\alpha)$. Furthermore, the symmetry condition (SP.0.i) implies that all even-order partial derivatives of f with respect to x must vanish at $x = 0$, for any α:

and that $\tilde{f}$ is even in x.

Now suppose, for some specific parameter value α_0, the equilibrium $x = 0$ is nonhyperbolic:

$$(\textit{bifurcation}), \qquad \text{(SP.0.ii)}$$

then expanding about $x = 0$ for any α, we have

$$f(x,\alpha) = x\,\tilde{f}(x,\alpha) = x\,[\,f_x(0,\alpha) + \tfrac{1}{6}\,f_{xxx}(0,\alpha)\,x^2 + O(|x|^4)\,] \qquad \text{(odd in } x\text{)},$$

and then expanding in α about α_0 and using (SP.0.ii), We assume two (generic) conditions hold, that the leading-order coefficients in the above expansions do not vanish:

$$a = f_{x\alpha}(0, \alpha_0) \neq 0 \qquad (\textit{transversality}), \tag{SP.1}$$

$$b = \tfrac{1}{6} f_{xxx}(0, \alpha_0) \neq 0 \qquad (\textit{nondegeneracy}). \tag{SP.2}$$

Then the family has a **symmetric pitchfork** bifurcation at $(0, \alpha_0)$. To simplify notation, let us assume

$$\alpha_0 = 0$$

so we have

and we can solve for equilibria,

One branch of equilibria, $x = p^0_{[1]}(\alpha) \equiv 0$, is already known as a consequence of the symmetry, and solving $\tilde{f}(x, \alpha) = 0$ (as we did for the fold bifurcation using the implicit function theorem, etc.), we obtain two more branches of nonzero equilibria $p^0_{[j]}(\alpha)$, $j = 2, 3$, for all α on one side of α_0 (which side, $\alpha < \alpha_0$ or $\alpha > \alpha_0$, is called the "direction" of the bifurcation), for all α sufficiently near α_0.

Once again, it can be proved that *certain* higher-order terms can be "transformed away" by some suitable change of variables (local homeomorphisms).

Theorem 3.2. *If* $f : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ *is* C^5 *in an open set containing* $(0, \alpha_0)$ *and satisfies the four conditions* (SP.0.i)–(SP.2)*, then the family of vector fields*

$$\frac{dx}{dt} = f(x, \alpha) \quad at \;\; (0, \alpha_0)$$

has a symmetric pitchfork bifurcation, locally topologically equivalent to the normal form

$$\frac{dy}{ds} = a\,\beta\,y + b\,y^3 \quad at \;\; (0, 0).$$

The topological normal form is

$$\frac{d\eta}{ds} = \gamma\,\eta\, \pm \eta^3.$$

Discussion of proof.

(1) Local analysis of equilibria. Solve

$$f(x, \alpha) = x\, \tilde{f}(x, \alpha) = 0.$$

In general, the implicit function theorem can be used to solve $\tilde{f}(x, \alpha) = 0$ (at least locally).

(2) Determine dynamics. For 1-dimensional vector fields, it is enough to determine the sign of $f(x, \alpha)$.

E.g. if $a > 0$, $b < 0$

Exercise. Draw local branching and bifurcation diagrams for the other three cases of $a \neq 0$, $b \neq 0$.

(3) Form invertible changes of parameter and state variables that "transform away" the higher-order terms in the Taylor expansion.

$$\beta = p(\alpha) = \alpha - \alpha_0 + o(|\alpha - \alpha_0|),$$

$$y = h(x, \alpha) = x - p_0^0 + o(|\alpha - \alpha_0| + |x - p_0^0|).$$

It is easy to verify that the stability of the bifurcating nonzero equilibria depends on the sign of b. If $b < 0$ (and therefore the bifurcating pair of nonzero equilibria are stable), the pitchfork bifurcation is called **supercritical**; if $b > 0$ (and the bifurcating pair of nonzero equilibria are unstable) the pitchfork bifurcation is called **subcritical**.

In applications, another common type of bifurcation for vector fields is the **transcritical** bifurcation (see HW3, problem 3).

The fold bifurcation for 1-dimensional maps

Let us now consider a 1-parameter family of 1-dimensional maps

$$x \mapsto f(x, \alpha), \qquad x \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1, \tag{‡}$$

and suppose it has, for some critical parameter value α_0, a nonhyperbolic fixed point p_0^0. In a 1-dimensional state space, this can occur only if $|\mu_0| = 1$, where the derivative $\mu_0 = f_x(p_0^0, \alpha_0) \in \mathbb{R}^1$ can be viewed as the multiplier (eigenvalue) of the 1×1 linearization, for $\alpha = \alpha_0$

$$\xi \mapsto f_x(p_0^0, \alpha_0)\, \xi, \qquad \xi \in \mathbb{R}^1.$$

With nonhyperbolic fixed points for 1-dimensional maps, there are two cases, $\mu_0 = +1$ and $\mu_0 = -1$. Here we first consider the easier case $\mu_0 = +1$.

If $f(x, \alpha)$ satisfies

$$f(p_0^0, \alpha_0) = p_0^0 \qquad (\textit{fixed point}), \tag{FM.0.i}$$

$$\mu_0 = f_x(p_0^0, \alpha_0) = 1 \qquad (\textit{bifurcation}), \tag{FM.0.ii}$$

$$a = f_\alpha(p_0^0, \alpha_0) \neq 0 \qquad (\textit{transversality}), \tag{FM.1}$$

$$b = \tfrac{1}{2} f_{xx}(p_0^0, \alpha_0) \neq 0 \qquad (\textit{nondegeneracy}), \tag{FM.2}$$

then the family (‡) has a **fold** bifurcation for maps. The Taylor series of the family (‡) at (p_0^0, α_0) is

and it can be proved that the specified higher-order terms do not qualitatively affect the local dynamics, in the sense that there exist changes of variables (local homeomorphisms of parameter and state variables) that "transform away" the higher-order terms. (Constructing the changes of variables is not as easy as in the vector field case – our textbook omits the details.)

Theorem 3.3. *If* $f : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ *is* C^3 *in an open set containing* (p_0^0, α_0) *and satisfies the four conditions* (FM.0.i)–(FM.2), *then the family of maps*

$$x \mapsto f(x, \alpha) \quad \textit{at} \;\; (p_0^0, \alpha_0)$$

has a fold bifurcation, locally topologically equivalent to the normal form

$$y \mapsto y + a\,\beta + b\,y^2 \quad \textit{at} \;\; (0, 0).$$

The topological normal form is

$$y \mapsto y + \beta \pm y^2.$$

Local dynamics of $y \mapsto y + a\beta + by^2$ at $(0,0)$, in the case $a > 0$, $b < 0$:

Exercise. Study the other three cases of $a \neq 0$, $b \neq 0$ for the normal form.

Families of maps (‡) can also have **transcritical** bifurcations and **symmetric pitchfork** bifurcations associated with a critical multiplier of the linearization $\mu_0 = +1$. Like the fold bifurcations for 1-dimensional maps, the transcritical and symmetric pitchfork bifurcations for 1-dimensional maps behave like discrete-time analogues of the corresponding bifurcations for 1-dimensional vector fields (no orientation reversals).