MATH 552 (2023W1) Lecture 18: Fri Oct 20

[**Last lecture:** Symmetric pitchfork bifurcation, Theorem 3.2. Fold bifurcation for maps, Theorem 3.3 ...]

$$x \mapsto f(x,\alpha), \qquad x \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1, \tag{1}$$

If $f(x,\alpha)$ satisfies

$$f(p_0^0, \alpha_0) = p_0^0 \qquad (\textit{fixed point}), \tag{FM.0.i}$$

$$\mu_0 = f_x(p_0^0, \alpha_0) = 1 \qquad (\textit{bifurcation}), \tag{FM.0.ii}$$

$$a = f_\alpha(p_0^0, \alpha_0) \neq 0 \qquad (\textit{transversality}), \tag{FM.1}$$

$$b = \tfrac{1}{2} f_{xx}(p_0^0, \alpha_0) \neq 0 \qquad (\textit{nondegeneracy}), \tag{FM.2}$$

then the family (‡) has a **fold** bifurcation for maps.

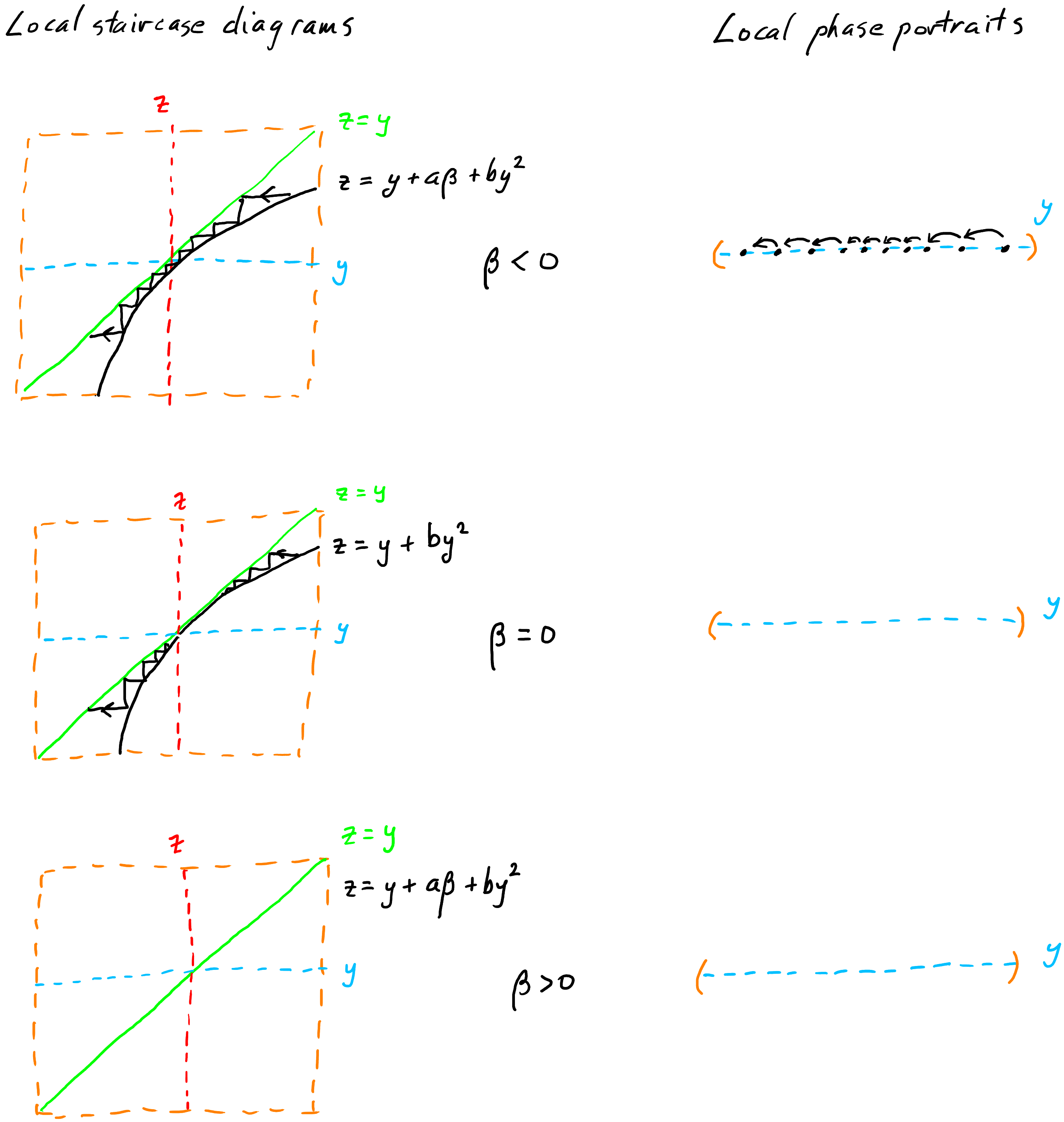
Theorem 3.3. *If $f : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ is C^3 in an open set containing (p_0^0, α_0) and satisfies the four conditions* (FM.0.i)–(FM.2), *then the family of maps*

$$x \mapsto f(x,\alpha) \quad \textit{at} \ (p_0^0, \alpha_0)$$

has a fold bifurcation, locally topologically equivalent to the normal form

$$y \mapsto y + a\,\beta + b\,y^2 \quad \textit{at} \ (0,0).$$

Local dynamics of $y \mapsto y + a\beta + by^2$ at $(0, 0)$, in the case $a > 0$, $b < 0$:



Exercise. Study the other three cases of $a \neq 0$, $b \neq 0$ for the normal form.

Families of maps (‡) can also have **transcritical** bifurcations and **symmetric pitchfork** bifurcations associated with a critical multiplier of the linearization $\mu_0 = +1$. Like the fold bifurcations for 1-dimensional maps, the transcritical and symmetric pitchfork bifurcations for 1-dimensional maps behave essentially like discrete-time analogues of the corresponding bifurcations for 1-dimensional vector fields.

In the case $\mu_0 = -1$, the associated generic bifurcation for a 1-dimensional map has no continuous-time analogue in a 1-dimensional vector field.

<u>The flip bifurcation for 1-dimensional maps</u>

Here we consider the resulting bifurcation in a 1-parameter family of 1-dimensional maps when the critical multiplier of the linearization at the nonhyperbolic fixed point is $\mu_0 = -1$. Consider again a family

$$x \mapsto f(x, \alpha), \qquad x \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1, \tag{‡}$$

where f is sufficiently smooth near a point $(p_0^0, \alpha_0) \in \mathbb{R}^1 \times \mathbb{R}^1$. If $f(x, \alpha)$ satisfies

$$(\textit{fixed point}), \qquad \text{(PD.0.i)}$$

$$(\textit{bifurcation}), \qquad \text{(PD.0.ii)}$$

then $x = p_0^0$ is a fixed point for $\alpha = \alpha_0$, and it is nonhyperbolic. In this case the implicit function theorem *can* be used (even if the fixed point is nonhyperbolic – **Exercise**) to solve

to obtain a locally unique smooth solution $x = p^0(\alpha)$ with $p^0(\alpha_0) = p_0^0$, a smooth curve $(\alpha, p^0(\alpha))$ of fixed points (i.e.

$$f(p^0(\alpha), \alpha) = p^0(\alpha)$$

for all α near α_0) through (α_0, p_0^0), so the number of fixed points does *not* change locally if we vary α near α_0. (It turns out though, that generically its topological type changes.)

For $\alpha = \alpha_0$ the linearization at p_0^0, namely $u \mapsto \mu_0 u = -u$, has 2-cycles, so we are going to look for 2-cycles in the nonlinear map for α near α_0. To make this (ultimately) easier, we first make coordinate changes, that do not change the dynamics but will eventually make explicit calculations possible. We first change coordinates with a shift

$$x = p^0(\alpha) + u \tag{I}$$

so that for any α (near α_0), $u = 0$ corresponds to the fixed point $x = p^0(\alpha)$ and the family (‡) is transformed

and becomes

(‡.I.a)

where the α-dependent coefficients are

$\mu(\alpha) =$

$\hat{f}_2(\alpha) =$

$\hat{f}_3(\alpha) =$

Note that (‡.I.a) has fixed point $u = 0$, i.e. $\hat{f}(0, \alpha) = 0$, for all α near α_0.

We assume a (generic) transversality condition

$$-a = \qquad (*)$$

which implies that the linearized stability of the fixed point $p^0(\alpha)$ changes, as α increases through α_0, because the value of the multiplier $\mu(\alpha)$ passes through $\mu(\alpha_0) = -1$ with nonzero "speed" with respect to α, i.e. transversally. Using an equivalent (**Exercise**) but more useful (in applications) expression for $\mu'(\alpha_0)$, we assume

$$-a = \qquad (\textit{transversality}). \qquad \text{(PD.1)}$$

We write (‡.I.a) as

where the α-dependent coefficients have Taylor expansions

$$\mu(\alpha) =$$

$$\hat{f}_2(\alpha) =$$

$$\hat{f}_3(\alpha) =$$

Now we write (‡.I.a) as

and make another coordinate change to "simplify" the Taylor expansion. A smooth "near-identity" coordinate change of the form

$$u = \qquad \text{(II)}$$

can be found, with a *specific choice* of α-dependent coefficient $h_2(\alpha)$ (**Exercise**, see HW4), so that in the transformed family the *linearization is unchanged*, and the quadratic Taylor coefficient in the state variable (i.e. the coefficient of v^2) is "removed" for all α, and the transformed family then has the form

$$v \mapsto \qquad \text{(‡.II.a)}$$

If this *specific choice* of $h_2(\alpha)$ is made, then it turns out (**Exercise**, see HW4) that the new cubic coefficient $g_3(\alpha)$, in the transformed family, becomes, at leading order in α,

$$g_3(\alpha_0) = \hat{f}_3(\alpha_0) + \left[\hat{f}_2(\alpha_0)\right]^2 \quad = -\,b.$$

We now assume that $g_3(\alpha_0) = -b$ does not vanish, but we express it in terms of the original family of maps (‡):

$$-\,b = \qquad (nondegeneracy) \tag{PD.2}$$

Thus the family of maps (‡) is locally smoothly conjugate (and therefore locally topologically equivalent) to

(‡.II.b)

It then can be proved that the higher-order terms in (‡.II.b) can be "transformed away" with some suitable changes of parameter and state variables (homeomorphisms)

$$\beta = p(\alpha) = \alpha - \alpha_0 + \cdots\ ,$$

$$y = h(v, \alpha) = v + \cdots\ ,$$

and as a consequence of the analysis of the normal form, the family of maps (‡) has a **flip** bifurcation (or **period doubling** bifurcation) at (p_0^0, α_0):

Theorem 3.4. *If $f : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ is C^4 in an open set containing (p_0^0, α_0) and satisfies the four conditions* (PD.0.i)–(PD.2)*, then the family*

$$x \mapsto f(x, \alpha) \quad at \;\; (p_0^0, \alpha_0)$$

has a flip (or period doubling) bifurcation, locally topologically equivalent to the normal form

$$y \mapsto -y - a\,\beta y - b\,y^3 \quad at \;\; (0, 0).$$

The topological normal form is

$$\eta \mapsto -\eta - \gamma\,\eta \mp \eta^3.$$

A bit more work is needed to show the existence and stability of bifurcating 2-cycles. The *second iterate* of (‡),

$$x \mapsto f^2(x, \alpha) = f(f(x, \alpha), \alpha)$$

is locally topologically equivalent to the second iterate of the normal form which can be seen, by explicit calculation (**Exercise**), to have bifurcating nonzero fixed points. These nonzero fixed points for the second iterate correspond to points on the orbits of nontrivial 2-cycles for the normal form. Their stability depends on the sign of b.