MATH 552 (2023W1) Lecture 19: Mon Oct 23

[**Last lecture:** ... fold bifurcation for maps, Flip bifurcation for maps ...]

$$x \mapsto f(x,\alpha), \qquad x \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1, \tag{‡}$$

where f is sufficiently smooth near a point $(p_0^0, \alpha_0) \in \mathbb{R}^1 \times \mathbb{R}^1$. Assume (‡) satisfies

$$f(p_0^0, \alpha_0) = p_0^0 \qquad (\textit{fixed point}), \tag{PD.0.i}$$

$$\mu_0 = f_x(p_0^0, \alpha_0) = -1 \qquad (\textit{bifurcation}). \tag{PD.0.ii}$$

Then, by the implicit function theorem, there is a smooth branch $x = p_0^0(\alpha)$ of fixed points,

$$f(p^0(\alpha), \alpha) = p^0(\alpha)$$

with $p^0(\alpha_0) = p_0^0$, for all α sufficiently near α_0.

First change of coordinates (a "shift")

$$x = p^0(\alpha) + u \tag{I}$$

so that for any α (near α_0), $u = 0$ corresponds to the fixed point $x = p^0(\alpha)$ and the family (‡) is transformed into

$$u \mapsto \hat{f}(u,\alpha) = \mu(\alpha)\, u + \hat{f}_2(\alpha)\, u^2 + \hat{f}_3(\alpha)\, u^3 + O(|u|^4), \tag{‡.I}$$

where

$$\mu(\alpha) = -[1 + a(\alpha - \alpha_0) + O(|\alpha - \alpha_0|^2)].$$

and we assume

$$-a = f_{x\alpha}(p_0^0, \alpha_0) + \tfrac{1}{2}\, f_{xx}(p_0^0, \alpha_0)\, f_\alpha(p_0^0, \alpha_0) \neq 0 \qquad (\textit{transversality}). \tag{PD.1}$$

We make a second coordinate change to "simplify" the Taylor expansion. A smooth "near-identity" coordinate change of the form

$$u = v + h^{(2)}(\alpha)\, v^2, \tag{II}$$

with a *specific choice* of α-dependent coefficient $h_2(\alpha)$ (**Exercise**, see HW4), so that in the transformed family the *linearization is unchanged*, and the quadratic Taylor coefficient in the state variable (i.e. the coefficient of v^2) is "removed" for all α, and the transformed (locally topologically equivalent) family then has the form

$$v \mapsto -v - a(\alpha - \alpha_0)v - bv^3 + O(|\alpha - \alpha_0|^2|v| + |\alpha - \alpha_0||v|^3 + |v|^4). \tag{‡.II.b}$$

where we assume

$$-b = \tfrac{1}{6} f_{xxx}(p_0^0, \alpha_0) + \left[\tfrac{1}{2} f_{xx}(p_0^0, \alpha_0)\right]^2 \neq 0 \qquad (\textit{nondegeneracy}) \tag{PD.2}$$

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It then can be proved that the higher-order terms in (‡.II.b) can be "transformed away" with some suitable changes of parameter and state variables (homeomorphisms)

$$\beta = p(\alpha) = \alpha - \alpha_0 + \cdots ,$$

$$y = h(v, \alpha) = v + \cdots ,$$

and (as a consequence of the analysis of the normal form) the family of maps (‡) can be shown to have a **flip** bifurcation (or **period doubling** bifurcation) at (p_0^0, α_0):

Theorem 3.4. *If* $f : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ *is* C^4 *in an open set containing* (p_0^0, α_0) *and satisfies the four conditions* (PD.0.i)–(PD.2)*, then the family*

$$x \mapsto f(x, \alpha) \quad \textit{at} \;\; (p_0^0, \alpha_0)$$

has a flip (or period doubling) bifurcation, locally topologically equivalent to the normal form

$$y \mapsto -y - a\,\beta y - b\,y^3 \quad \textit{at} \;\; (0, 0).$$

The topological normal form is

$$\eta \mapsto -\eta - \gamma\,\eta \mp \eta^3.$$

A bit more work is needed to show the existence and stability of bifurcating 2-cycles. The *second iterate* of (‡),

$$x \mapsto f^2(x, \alpha) = f(f(x, \alpha), \alpha)$$

is locally topologically equivalent to the second iterate of the normal form which can be seen, by explicit calculation (**Exercise**), to have bifurcating nonzero fixed points. These nonzero fixed points for the second iterate correspond to points on the orbits of nontrivial 2-cycles for the normal form. Their stability depends on the sign of b.

A subtle point: The normal form has symmetry, but the original family of maps (‡) in general does not. Symmetry is introduced into the *lower*-order terms with the transformation (II), then by "transforming away" higher-order terms we lost some information. This does not affect local topological equivalence, but is sometimes worth noting.

Local bifurcation diagram for the normal form if $a > 0$, $b < 0$

Typical local branching diagram for (‡) if $a > 0$, $b < 0$

Exercise. Draw diagrams for the other three cases of $a \neq 0$, $b \neq 0$.

Poincaré normal forms

For vector fields or maps, there is a systematic method to make nonlinear "near-identity" changes of state variables that "simplify" the lower-order terms of the Taylor expansion at an equilibrium or a fixed point. Furthermore, the method can be modified to apply to families of vector fields or maps. (In fact the method was used above, for a family of 1-dimensional maps at a flip bifurcation.) Here we develop the method for n-dimensional vector fields.

Consider a smooth vector field

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n,$$

and assume $f(0) = 0$. If $n = 2$, we have in component form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$$

where $f_1(0,0) = 0$ and $f_2(0,0) = 0$. A Taylor expansion about the origin $(0,0)$ has the form where the a_j are the elements of the derivative (2×2 matrix) $f_x(0)$

$$a_1 = \frac{\partial f_1}{\partial x_1}(0,0), \qquad a_2 = \frac{\partial f_1}{\partial x_2}(0,0),$$
$$a_3 = \frac{\partial f_2}{\partial x_1}(0,0), \qquad a_4 = \frac{\partial f_2}{\partial x_2}(0,0),$$

and the b_j are the second-order Taylor coefficients

$$b_1 = \frac{1}{2!\,0!}\,\frac{\partial^2 f_1}{\partial x_1^2}(0,0), \quad b_2 = \frac{1}{1!\,1!}\,\frac{\partial^2 f_1}{\partial x_1 \partial x_2}(0,0), \quad b_3 = \frac{1}{0!\,2!}\,\frac{\partial^2 f_1}{\partial x_2^2}(0,0),$$
$$b_4 = \frac{1}{2!\,0!}\,\frac{\partial^2 f_2}{\partial x_1^2}(0,0), \quad b_5 = \frac{1}{1!\,1!}\,\frac{\partial^2 f_2}{\partial x_1 \partial x_2}(0,0), \quad b_6 = \frac{1}{0!\,2!}\,\frac{\partial^2 f_6}{\partial x_2^2}(0,0).$$

The Taylor expansion can be written more concisely as

$$\dot{x} = A\,x + f^{(2)}(x) + O(\|x\|^3), \quad x \in \mathbb{R}^2$$

where $A = f_x(0)$ is the (Jacobian) matrix of first order partial derivatives evaluated at 0 and $f^{(2)}(x)$ denotes the vector field of second-order terms in the Taylor expansion about 0.

Notice that $f^{(2)}(x)$ belongs to the set, which we call H_2, of vector fields whose components are *homogeneous polynomials* in $x = (x_1, x_2)$ *of order* 2. Also notice we can write any such $f^{(2)}(x)$ as

$$f^{(2)}(x) = b_1 \begin{pmatrix} x_1^2 \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} x_1 x_2 \\ 0 \end{pmatrix} + \cdots + b_6 \begin{pmatrix} 0 \\ x_2^2 \end{pmatrix},$$

which shows that the set H_2 is actually a vector space, of dimension 6 (when $n = 2$), with basis

and with this basis we can represent $f^{(2)}(x) \in H_2$ by the vector

Similarly, in $\mathbb{R}^n$ with $n = 3, 4, \ldots,$ if f is C^{m+1} with $m = 2, 3, \ldots,$ by Taylor's theorem with remainder, we can make a Taylor expansion of

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \quad \text{with } f(0) = 0$$

at the equilibrium $0 \in \mathbb{R}^n$

$$\dot{x} = A\,x + f^{(2)}(x) + f^{(3)}(x) + \cdots + f^{(m)}(x) + O(\|x\|^{m+1}), \tag{3.3}$$

where $A\,x = f_x(0)\,x$ is the linearization of the vector field at the equilibrium, and for $k = 2, 3, \ldots, m$ each $f^{(k)}$ belongs to H_k, the finitedimensional vector space of all vector fields from $\mathbb{R}^n$ into $\mathbb{R}^n$ whose components are homogeneous polynomials of order k.

(To simplify Poincaré normal form calculations in practice, it is usual to choose coordinates x so that A is in real or Jordan normal form.) We start by "simplifying" the order $k = 2$ terms. Introduce a "near-identity" coordinate change (a local diffeomorphism) of the form

$$x = y + h^{(2)}(y)$$

where $h^{(2)} \in H_2$. E.g. if $n = 2$

Applying any such coordinate change, we get

then solving for $\dot{y}$

and expanding