MATH 552 (2023W1) Lecture 20: Wed Oct 25

[**Last lecture:** ... flip (period doubling) bifurcation for maps. Poincaré normal forms ...]

Consider

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n,$$

with

$$f(0) = 0.$$

E.g. if $n = 2$, Taylor expansion about $x = 0$ is

$$\dot{x} = A\,x + f^{(2)}(x) + O(\|x\|^3), \quad x \in \mathbb{R}^2$$

where $A = f_x(0)$ is the (Jacobian) matrix of first order partial derivatives evaluated at 0 and $f^{(2)}(x)$ denotes the vector field of second-order terms in the Taylor expansion about 0,

$$f^{(2)}(x) = \begin{pmatrix} b_1\,x_1^2 + b_2\,x_1 x_2 + b_3\,x_2^2 \\ b_4\,x_1^2 + b_5\,x_1 x_2 + b_6\,x_2^2 \end{pmatrix}.$$

We see that $f^{(2)}(x)$ belongs to the set, which we call H_2, of vector fields whose components are *homogeneous polynomials* in $x = (x_1, x_2)$ *of order* 2.

=====

We can write any such $f^{(2)}(x)$ as

$$f^{(2)}(x) = b_1 \begin{pmatrix} x_1^2 \\ \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} x_1 x_2 \\ \\ 0 \end{pmatrix} + \cdots + b_6 \begin{pmatrix} 0 \\ \\ x_2^2 \end{pmatrix},$$

which shows that the set H_2 is actually a vector space, of dimension 6 (when $n = 2$), with basis

and with this basis we can represent $f^{(2)}(x) \in H_2$ by the vector

Similarly, in $\mathbb{R}^n$ with $n = 3, 4, \ldots$, if f is C^{m+1} with $m = 2, 3, \ldots$, by Taylor's theorem with remainder, we can make a Taylor expansion of

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \quad \text{with } f(0) = 0$$

at the equilibrium $0 \in \mathbb{R}^n$

$$\dot{x} = A\,x + f^{(2)}(x) + f^{(3)}(x) + \cdots + f^{(m)}(x) + O(\|x\|^{m+1}), \tag{3.3}$$

where $A\,x = f_x(0)\,x$ is the linearization of the vector field at the equilibrium, and for $k = 2, 3, \ldots, m$ each $f^{(k)}$ belongs to H_k, the finite-dimensional vector space of all vector fields from $\mathbb{R}^n$ into $\mathbb{R}^n$ whose components are homogeneous polynomials of order k.

(To simplify Poincaré normal form calculations in practice, it is usual to choose coordinates x so that A is in real or Jordan normal form.) We start by "simplifying" the order $k = 2$ terms. Introduce a "near-identity" coordinate change (a local diffeomorphism) of the form

$$x = y + h^{(2)}(y)$$

where $h^{(2)} \in H_2$. E.g. if $n = 2$

Applying any such coordinate change, we get

then solving for $\dot{y}$

and expanding

and expanding further, keeping explicitly all terms up to order 2

we see the linearization is unchanged in the new coordinates and the vector field (3.3) is transformed into the (locally smoothly equivalent) vector field

$$\dot{y} = Ay - (L_A h^{(2)})(y) + f^{(2)}(y) + O(\|y\|^3),$$

where for any integer $k \geq 2$, $L_A : H_k \to H_k$ is defined by

$$(L_A h^{(k)})(y) = h_y^{(k)}(y)Ay - Ah^{(k)}(y).$$

Exercise. Verify that $L_A : H_k \to H_k$, and is a *linear operator* (we can find a matrix representation of L_A with respect to any basis for H_k).

For $k = 2$, we find the range $L_A(H_2)$ and choose a *complementary subspace* $\tilde{H}_2$ in H_2 (not necessarily unique, and not always useful to take the orthogonal complement of $L_A(H_2)$) so that

$$H_2 = L_A(H_2) \oplus \tilde{H}_2.$$

Relative to any *specific* complementary subspace $\tilde{H}_2$, every $f^{(2)} \in H_2$ has a unique decomposition

$$f^{(2)}(y) = g^{(2)}(y) + r^{(2)}(y), \qquad g^{(2)} \in L_A(H_2), \quad r^{(2)} \in \tilde{H}_2.$$

Then since $g^{(2)}$ belongs to the range $L_A(H_2)$, there exists some *specific* $h^{(2)} \in H_2$ such that the coordinate change "removes" this component of $f^{(2)}$ that lies in the range,

and therefore transforms (3.3) into

$$\dot{y} = A\,y + r^{(2)}(y) + O(\|y\|^3). \tag{3.4}$$

In this way the vector field (3.3) is "simplified" into (3.4) by a coordinate change that "removes" as many coefficients of order 2 as possible. The term $r^{(2)}(y)$ (it might be zero) is said to contain the **resonant terms of order** 2, and we say the vector field has been put into **Poincaré normal form up to order** 2. We also say that (3.4) is the Poincaré normal form of (3.3) up to order 2.

By induction, one can show that as long as f is smooth enough, the vector field (3.3) can be transformed into Poincaré normal form up to any finite order m, $m \geq 2$.

Theorem 3.5. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^{m+1} $(m \geq 2)$ in an open set containing 0 and $f(0) = 0$, then there exists a coordinate change*

$$x = y + h^{(2)}(y) + \cdots + h^{(m)}(y), \quad h^{(k)} \in H_k, \quad k = 2, \cdots, m,$$

that transforms the Taylor expansion of f at the equilibrium 0, up to order m

$$\dot{x} = A\,x + f^{(2)}(x) + \cdots + f^{(m)}(x) + O(\|x\|^{m+1}), \quad f^k \in H_k, \quad k = 2, \cdots, m,$$

into the (locally smoothly equivalent) Poincaré normal form, up to order m

$$\dot{y} = A\,y + r^{(2)}(y) + \cdots + r^{(m)}(y) + O(\|y\|^{m+1}), \quad r^{(k)} \in \tilde{H}_k, \quad k = 2, \cdots, m,$$

where each $r^{(k)}$ contains the resonant terms of order k, $k = 2, \cdots, m$.

A similar method exists for n-dimensional maps (e.g. HW4 problem 2(b), $n = 1$). Then the methods can be modified slightly for families of n-dimensional vector fields or maps (e.g. HW4 problem 2(c)).

Example 3.B. (This example is used later in the analysis of the Hopf bifurcation.) Consider a 2-dimensional vector field

$$\dot{x} = f(x), \quad x \in \mathbb{R}^2. \tag{3.B.0}$$

We assume f is smooth (C^4 will turn out to be enough), and assume $p^0 \in \mathbb{R}^2$ is an equilibrium, i.e.

$$f(p^0) = 0.$$

In addition, we assume that the 2×2 matrix of the linearization

$$A = f_x(p^0) \quad \text{has eigenvalues } \lambda_1 = i\omega,\ \lambda_2 = -i\omega,$$

where $\omega > 0$. So the equilibrium p^0 is nonhyperbolic, a so-called "Hopf point" or "linear centre". We will analyze the *nonlinear* dynamics near this equilibrium using Poincaré normal form theory.

But first, we prepare: we make an initial coordinate change, a shift

$$x = p^0 + u \tag{I}$$

that transforms (3.B.0) into

$$\dot{u} = \hat{f}(u) = A\,u + \hat{f}^{(2)}(u) + \hat{f}^{(3)}(u) + O(\|u\|^4), \quad \hat{f}^{(k)} \in H_k, \tag{3.B.1}$$

where $\hat{f}(u) = f(p^0 + u) - f(p^0) = f(p^0 + u)$.

Next we transform the entire nonlinear equation so the linear part is in real normal form. Find an eigenvector $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \in \mathbb{C}^2$, $Aq = i\omega q$, and make the coordinate change

This transforms (3.B.1) into

where

$$R = T^{-1} A T = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

It turns out the Poincaré normal form calculations will be simpler if the linear part is in Jordan normal form, which is diagonal. So we complexify (think of $v \in \mathbb{C}^2$) and make the coordinate change that transforms (3.B.2) into the form

where

$$J = U^{-1}RU = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (\lambda_1 = i\omega,\ \lambda_2 = -i\omega)$$

Now (after some preparation) we use the Poincaré normal form theory. By Theorem 3.5, *there exists* a coordinate change (diffeomorphism)

that transforms (3.B.3) into the (smoothly equivalent) Poincaré normal form up to order 3 (or the "cubic normal form")

We find the resonant terms $r^{(2)}$ and $r^{(3)}$ explicitly, starting with $r^{(2)}$. A convenient basis for the vector space of second order terms H_2 is

and dim $H_2 = 6$.