MATH 552 (2023W1) Lecture 21: Fri Oct 27

[**Last lecture:** ... Poincaré normal forms ...]

Theorem 3.5. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^{m+1} ($m \geq 2$) in an open set containing 0 and $f(0) = 0$, then there exists a coordinate change*

$$x = y + h^{(2)}(y) + \cdots + h^{(m)}(y), \quad h^{(k)} \in H_k, \quad k = 2, \cdots, m,$$

that transforms the Taylor expansion of f at the equilibrium 0, up to order m

$$\dot{x} = A\,x + f^{(2)}(x) + \cdots + f^{(m)}(x) + O(\|x\|^{m+1}), \quad f^k \in H_k, \quad k = 2, \cdots, m,$$

into the (locally smoothly equivalent) Poincaré normal form, up to order m

$$\dot{y} = A\,y + r^{(2)}(y) + \cdots + r^{(m)}(y) + O(\|y\|^{m+1}), \quad r^{(k)} \in \tilde{H}_k, \quad k = 2, \cdots, m,$$

where each $r^{(k)}$ contains the resonant terms of order k, $k = 2, \cdots, m$.

Example 3.B, continued (used later in the analysis of the **Hopf bifurcation**).

$$\dot{x} = f(x), \quad x \in \mathbb{R}^2, \tag{3.B.0}$$

where

$$f(p^0) = 0.$$

We assume that the 2×2 matrix of the linearization

$$A = f_x(p^0) \quad \text{has eigenvalues } \lambda_1 = i\omega,\ \lambda_2 = -i\omega,$$

where $\omega > 0$. So the equilibrium p^0 is nonhyperbolic, a so-called "Hopf point" or "linear centre".

An initial coordinate change, a shift

$$x = p^0 + u \tag{I}$$

that transforms (3.B.0) into

$$\dot{u} = \hat{f}(u) = A\,u + \hat{f}^{(2)}(u) + \hat{f}^{(3)}(u) + O(\|u\|^4), \quad \hat{f}^{(k)} \in H_k, \tag{3.B.1}$$

where $\hat{f}(u) = f(p^0 + u) - f(p^0) = f(p^0 + u)$.

Next, coordinate changes (II) and (III) transform the entire nonlinear equation so the linear part is in Jordan normal form,

$$\dot{z} = J\,z + g^{(2)}(z) + g^{(3)}(z) + O(\|z\|^4), \tag{3.B.3}$$

where

$$J = U^{-1} R U = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (\lambda_1 = i\omega,\ \lambda_2 = -i\omega)$$

Now, by Theorem 3.5 (Poincaré normal form theory), *there exists* a coordinate change (a local diffeomorphism) (IV) that transforms (3.B.3) into the (smoothly equivalent) Poincaré normal form up to order 3 (or the "cubic normal form")

$$\dot{\zeta} = J\,\zeta + r^{(2)}(\zeta) + r^{(3)}(\zeta) + O(\|\zeta\|^4) \qquad \text{(3.B.4a)}$$

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We find the resonant terms $r^{(2)}$ and $r^{(3)}$ explicitly, starting with $r^{(2)}$. A convenient basis for the vector space of second order terms H_2 is

and dim $H_2 = 6$. We find the 6×6 matrix representation $M_J^{(2)}$ of $L_J : H_2 \to H_2$. Recall that if $\mathcal{B}^{(2)} = \{e_1^{(2)}, e_2^{(2)}, \ldots, e_6^{(2)}\}$ is a basis, and if

then the j-th **column** of the matrix representation $M_J^{(2)}$ with respect to the basis $\mathcal{B}^{(2)}$ is

$$\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{6j} \end{pmatrix}$$

We calculate

i.e. $L_J e_1^{(2)} = \lambda_1 \, e_1^{(2)}$, so column 1 of $M_J^{(2)}$ is

Exercise.

Thus $M_J^{(2)}$ is a diagonal 6×6 matrix and since $\lambda_1 = i\omega$, $\lambda_2 = -i\omega$, $\omega > 0$ it is easy to see that $M_J^{(2)}$ is nonsingular, with nullity$(M_J^{(2)}) = 0$, rank$(M_J^{(2)}) = 6$ (full rank). This means

$$L_J : H_2 \to H_2 \quad \text{is onto, i.e. } L_J(H_2) = H_2, \quad \tilde{H}_2 = \{0\}, \quad r^{(2)} = 0,$$

and there is some *specific* coordinate change (**Exercise:** find explicitly what $h^{(2)}(\zeta)$ does this) that "removes" all second order terms.

Then we find $r^{(3)}$. We use the ordered basis $\mathcal{B}^{(3)}$ given by

for H_3, $\dim(H_3) = 8$, and find the matrix representation $M_J^{(3)}$ of $L_J : H_3 \to H_3$ with respect to $\mathcal{B}^{(3)}$ (**Exercise**) is the 8×8 matrix, again diagonal

$$M_J^{(3)} = \mathrm{diag}(\, 2\lambda_1,\, 0,\, 2\lambda_2,\, 3\lambda_2 - \lambda_1,\, 3\lambda_1 - \lambda_2,\, 2\lambda_1,\, 0,\, 2\lambda_2\,)$$

thus $M_J^{(3)}$ is singular, with nullity$(M_J^{(3)}) = 2$, and rank$(M_J^{(3)}) = 6$ which is deficient by 2. This implies

$$\dim(L_J(H_3)) = 6, \quad \dim(\tilde{H}_3) = 2.$$

Since $M_J^{(3)}$ is conveniently diagonal, it is easy to characterize the range

$L_J(H_3)$ in terms of the basis elements as

$$L_J(H_3) = \mathrm{span}\{\, e_1^{(3)},\, e_3^{(3)},\, e_4^{(3)},\, e_5^{(3)},\, e_6^{(3)},\, e_8^{(3)}\,\}$$

and it is easy to choose a complementary subspace as

$$\tilde{H}_3 = \mathrm{span}\{\, e_2^{(3)},\, e_7^{(3)}\,\}.$$

(It's not always this easy, see HW 4. Also, there are infinitely many valid choices for $\tilde{H}_3$, e.g.

but we try to make a choice that minimizes subsequent calculations.)

With this (initial) choice of complementary subspace, $r^{(3)}$ has the form

for some (complex) coeffecents c_1 and c_2. Then (3.B.4a) is, explicitly

using some *specifically chosen* $h^{(2)}$ and $h^{(3)}$ in (IV).

Now "realify" by putting $\zeta_2 = \bar{\zeta}_1$. Then the second component of

(3.B.4b) is the complex conjugate of the first component, thus $c_2 = \bar{c}_1$,

and now we only need to keep track of the first complex component. Then in polar coordinates

$$\zeta_1 = r\,e^{i\,\theta} \tag{V}$$

we get

or

[After the Poincaré normal form coordinate change (IV), the lower-order terms have circular symmetry (or "$\mathbb{S}^1$-equivariance"), and any coupling between r and θ has been "moved" to higher order (e.g. $O(r^4)$ in the $\dot{r}$ equation).]

If $b \neq 0$, then in (3.B.5b) we can see that

$$\begin{aligned} \operatorname{sign}(\dot{r}) &= \operatorname{sign}(b) \quad \text{for all sufficiently small } r > 0, \\ \operatorname{sign}(\dot{\theta}) = \operatorname{sign}(\omega) &= + \quad \text{for all sufficiently small } r > 0 \end{aligned}$$

and so it can be proved that the higher-order terms now do not affect the dynamics, in some sufficiently small open neighbourhood of the equilibrium. In fact, the higher-order terms can be "transformed away" by some suitable change of variables (homeomorphism) and so the original vector field (3.B.0) at $x = p^0$, is locally topologically equivalent to

$$\begin{aligned} \frac{dy_1}{ds} &= -\omega\, y_2 + b\,(y_1^2 + y_2^2)^2 y_1, \\ \frac{dy_2}{ds} &= \omega\, y_1 + b\,(y_1^2 + y_2^2)^2 y_2 \end{aligned} \tag{3.B.6}$$

at $(y_1, y_2) = (0, 0)$, which in polar coordinates $y_1 = \rho\cos(\phi)$, $y_2 = \rho\sin(\phi)$ is

$$\begin{aligned} \frac{d\rho}{ds} &= b\,\rho^3, \\ \frac{d\phi}{ds} &= \omega \end{aligned}$$

We need an efficient way to find sign(b) in a typical application. **Exercise:** what if $b = 0$?