MATH 552 (2023W1) Lecture 22: Mon Oct 30

[**Last lecture:** ... Poincaré normal forms ...]

Example 3.B ("Hopf point" normal form), summary.

$$\frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^2, \tag{3.B.0}$$

with an equilibrium,

$$f(p^0) = 0,$$

that is nonhyperbolic,

$$A = f_x(p^0) \quad \text{has eigenvalues } \lambda_{1,2} = \pm i\,\omega,\ \omega > 0,$$

a "Hopf point" equilibrium.

After five coordinate changes, we get the "cubic" Poincaré normal form (i.e. up to order 3) of (3.B.0), expressed in polar coordinates

$$\begin{aligned} \frac{dr}{dt} &= b\,r^3 + O(r^4), \\ \frac{d\theta}{dt} &= \omega + O(r^2) \end{aligned} \tag{3.B.5}$$

where b is some real number. Now a theorem says we can "ignore" the higher order terms and determine the correct dynamics, up to local topological equivalence. Stability or instability of the equilibrium depends on the sign of the cubic normal form coefficient b.

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Projection method for computation of the cubic normal form coefficient b

In Example 3.B, we skip transformations (II)–(III) with some linear algebra.

For two complex vectors

$$p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \in \mathbb{C}^2$$

we define their inner product (**note** our convention where to put the complex conjugation!) as

$$\langle\, p, q \,\rangle = \bar{p}_1 q_1 + \bar{p}_2 q_2.$$

If A is a 2×2 matrix of complex constants, we define its **adjoint** matrix A^* by

$$\langle\, A^* p, q \,\rangle = \langle\, p, Aq \,\rangle \quad \text{for all } p, q \in \mathbb{C}^2$$

Exercise. $A^* = \bar{A}^\intercal$ (and therefore $A^* = A^\intercal$ for a real matrix).

The projection method: first, find an eigenvector $q \in \mathbb{C}^2$, for the eigenvalue $\lambda_1 = i\,\omega$, $\omega > 0$

$$A\,q = \lambda_1\, q, \quad q \neq 0,$$

Then, find an **adjoint eigenvector** $p \in \mathbb{C}^2$,

$$A^*\, p = \bar{\lambda}_1\, p, \quad p \neq 0.$$

Exercise. It follows that

$$\langle\, p, \bar{q}\,\rangle = 0, \qquad \langle\, p, q\,\rangle \neq 0.$$

Normalize p so that

$$\langle\, p, q\,\rangle = 1.$$

Now, any $u \in \mathbb{R}^2$ is expressed uniquely as $u = z_1\, q + \bar{z}_1\, \bar{q}$, i.e.

$$\begin{pmatrix} u_1 \\ \\ u_2 \end{pmatrix} = z_1 \begin{pmatrix} q_1 \\ \\ q_2 \end{pmatrix} + \bar{z}_1 \begin{pmatrix} \bar{q}_1 \\ \\ \bar{q}_2 \end{pmatrix}$$

where (**Exercise**)

$$z_1 = \langle\, p, u\,\rangle$$

($z_1 q$ is the projection of u onto $\text{span}\{q\}$).

Put $u = z_1\, q + \bar{z}_1\, \bar{q}$ (i.e. $u_j = z_1 q_j + \bar{z}_1 \bar{q}_j,\ j = 1, 2$) in the Taylor expansion (3.B.1) to get

and then taking the inner product with the adjoint eigenvector p (i.e. taking the projection of the ODE (3.B.1) onto the eigenvector direction $\text{span}\{q\}$), recalling the exercise above and the normalization of p, we get

Exercise. (long!) Find explicitly the $h^{(2)}$, $h^{(3)}$ in (IV) that transforms (3.B.3) into (3.B.4), then determine the complex cubic normal form coefficient c_1 in terms of the complex coefficients g_{jk} in (3.B.3).

The result of this exercise is important:

$$\boxed{b = \mathrm{Re}(c_1) = \mathrm{Re}\left(\frac{1}{2}\, g_{21} + \frac{i}{2\omega}\, g_{20}\, g_{11}\right)} \qquad (**)$$

The Hopf bifurcation for 2-dimensional vector fields

A **Hopf bifurcation** is the generic bifurcation of limit cycles in a family of vector fields, where at a critical parameter value, the linearization at an equilibrium has a pair of purely imaginary eigenvalues i.e. an equilibrium is a "Hopf point".

Consider a smooth 1-parameter family of 2-dimensional vector fields

$$\dot{x} = f(x, \alpha), \qquad x \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}^1. \tag{3.4.0}$$

and assume there exist $p_0^0 \in \mathbb{R}^2$ and $\alpha_0 \in \mathbb{R}^1$ such that

$$f(p_0^0, \alpha_0) = 0 \qquad (\textit{equilibrium}), \tag{H.0.i}$$

and

$$A_0 = f_x(p_0^0, \alpha_0) \text{ has eigenvalues } \pm\, i\,\omega_0,\ \ \omega_0 > 0 \qquad (\textit{bifurcation}). \tag{H.0.ii}$$

(Equivalently, $\mathrm{tr}(A_0) = 0$ and $\det(A_0) = \omega_0^2 > 0$.) So for $\alpha = \alpha_0$, $x = p_0^0$ is an equilibrium that is a nonhyperbolic "Hopf point".

Exercise. Apply the implicit function theorem, to obtain a locally defined, locally unique, smooth curve $(p^0(\alpha), \alpha)$ of equilibria through (p_0^0, α_0).

For each α, linearize at the equilibrium $p^0(\alpha)$: the resulting 2×2 real matrix

$$A(\alpha) = f_x(p^0(\alpha), \alpha)$$

depends smoothly on α, and at $\alpha = \alpha_0$ the eigenvalues $\pm i\,\omega_0$ are simple, so for all α sufficiently near α_0, $A(\alpha)$ has eigenvalues

$$\lambda_1(\alpha) = \mu(\alpha) + i\omega(\alpha), \quad \lambda_2(\alpha) = \mu(\alpha) - i\omega(\alpha),$$

where $\mu(\alpha)$, $\omega(\alpha)$ are real-valued, smooth, and satisfy

$$\mu(\alpha_0) = 0, \quad \omega(\alpha_0) = \omega_0 > 0.$$

For computational convenience, recall

$$\mu(\alpha) = \mathrm{Re}(\lambda_1(\alpha)) = \tfrac{1}{2}\,\mathrm{tr}(A(\alpha)),$$

for all α sufficiently near α_0. We assume that for α increasing through α_0, the eigenvalues cross the imaginary axis with nonzero "speed":

$$a = \mu'(\alpha_0) = \tfrac{1}{2}\,\tfrac{d}{d\alpha}\,\mathrm{tr}(A(\alpha))\Big|_{\alpha=\alpha_0} \neq 0 \qquad (\textit{transversality}), \tag{H.1}$$

so that the linearized stability (and topological type) of the equilibrium $p^0(\alpha)$ changes, as α increases through α_0.

Now we begin a sequence of (α-dependent smooth families of) coordinate changes analogous to the coordinate changes we made in Example

3.B, in an effort to reduce the problem to one where explicit calculations are possible to determine the dynamics. Similar ideas to modify coordinate changes, for a system to a family of systems, are used in HW4 problem 2(b)(c). Make the first coordinate change (actually, a family of coordinate changes)

$$x = p^0(\alpha) + u \tag{I}$$

to transform the family (3.4.0) into the family

$$\dot{u} = \hat{f}(u, \alpha), \tag{3.4.1}$$

where

$$\hat{f}(u, \alpha) = f(p^0(\alpha) + u, \alpha) - f(p^0(\alpha), \alpha) = f(p^0(\alpha) + u, \alpha),$$

and the equilibrium is now $u = 0$ for all α near α_0, i.e. $\hat{f}(0, \alpha) = 0$. A Taylor expansion in u, up to order 3, has the form

where

is the family of 2×2 matrices or linearizations, and are the families of Taylor expansion terms of orders 2 and 3 in u, with α-dependent coefficients.

To find the cubic Poincaré normal form coefficient, we use the projection method again, but modified for families. For all α sufficiently near α_0 we can, *at least in principle*, find a smoothly parametrized family of complex eigenvectors $q(\alpha) \in \mathbb{C}^2$

$$A(\alpha)\, q(\alpha) = \lambda_1(\alpha)\, q(\alpha), \quad q(\alpha) \neq 0$$

and a smoothly parametrized family of adjoint eigenvectors $p(\alpha) \in \mathbb{C}^2$

$$A(\alpha)^{\mathsf{T}}\, p(\alpha) = \bar{\lambda}_1(\alpha)\, p(\alpha), \quad p(\alpha) \neq 0$$

normalized so that

$$\langle\, p(\alpha),\, q(\alpha)\,\rangle = 1$$

for all α sufficiently near α_0. (**NOTE:** See below, we normally need to do this *only for* $\alpha = \alpha_0$, *not* for all α in an interval.) Then take the projection of (3.4.1) onto the eigenvector (family) direction span$\{q(\alpha)\}$, by putting

$$u = z_1\, q(\alpha) + \bar{z}_1\, \bar{q}(\alpha), \quad u \in \mathbb{R}^2,\ z_1 \in \mathbb{C},$$

and taking the inner product with the adjoint eigenvector $p(\alpha)$, to obtain

Then, by Poincaré normal form theory (modified for families), there is a smoothly parametrized family of smooth coordinate changes

$$z_1 = \zeta_1 + h^{(2)}(\zeta_1, \bar{\zeta}_1, \alpha) + h^{(3)}(\zeta_1, \bar{\zeta}_1, \alpha), \quad h^{(k)}(\cdot, \cdot, \alpha) \in H_k, \;\; k = 2,\, 3, \tag{IV}$$

for all α sufficiently near α_0, that transforms (3.4.3) into its Poincaré normal form up to order 3

$$\dot{\zeta}_1 = \lambda_1(\alpha)\, \zeta_1 + c_1(\alpha)\, |\zeta_1|^2 \zeta_1 + O(|\zeta_1|^4), \quad \zeta_1 \in \mathbb{C},$$

In polar coordinates $\zeta_1 = r\, e^{i\theta}$ this is

Now expanding in α about α_0, the system becomes

If $\alpha = \alpha_0$, notice that (3.4.0) becomes Example 3.B, and we evaluate the cubic normal form coefficient b with formula $(**)$, Lecture 20. Since $(**)$ does not depend on any α-derivatives, we could set $\alpha = \alpha_0$ in (3.4.0) to calculate b (see below). If

$$b = \operatorname{Re} c_1(\alpha_0) = \operatorname{Re}\left[\frac{1}{2}\, g_{21}(\alpha_0) + \frac{i}{\omega_0}\,\frac{1}{2}\, g_{20}(\alpha_0)\, g_{11}(\alpha_0)\right] \neq 0 \qquad \text{(H.2)}$$

$$(nondegeneracy),$$

then the family (3.4.0) has a **Hopf bifurcation**:

Theorem 3.6. *If* $f : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2$ *is* C^4 *in an open set containing* (p_0^0, α_0) *and satisfies the four conditions* (H.0.i)–(H.2), *then the family*

$$\frac{dx}{dt} = f(x, \alpha) \quad at\ (p_0^0, \alpha_0)$$

has a Hopf bifurcation, locally topologically equivalent to the normal form

$$\begin{aligned} \frac{dy_1}{ds} &= a\,\beta\, y_1 - \omega_0\, y_2 + b\,(y_1^2 + y_2^2)\, y_1 \\ \frac{dy_2}{ds} &= \omega_0\, y_1 + a\,\beta\, y_2 + b\,(y_1^2 + y_2^2)\, y_2 \end{aligned} \qquad at\ ((0,0),0).$$

By rescaling variables and possibly changing the sign of β, we could obtain the topological normal form with $\omega_0 = 1$, $a = 1$, $b = \pm 1$.

In polar coordinates $y_1 = \rho \cos(\phi)$, $y_2 = \rho \sin(\phi)$, the normal form is

$$\begin{aligned} \frac{d\rho}{ds} &= a\beta\,\rho + b\,\rho^3, \\ \frac{d\phi}{ds} &= \omega_0, \end{aligned} \qquad (\rho, \phi) \in \mathbb{R}_+ \times \mathbb{S}^1,$$

which is used to easily determine the existence and stability of limit cycles (**Exercise**). One verifies there exist limit cycles bifurcating from the equilibrium at $\beta = 0$, which are stable or unstable depending on the sign of b. If $b < 0$ the Hopf bifurcation is called **supercritical**; if $b > 0$ the Hopf bifurcation is called **subcritical**.

E.g. if $a > 0$ and $b < 0$: