MATH 552 (2023W1) Lecture 23: Wed Nov 1

[**Last lecture:** Projection method. Hopf bifurcation ...]

$$\dot{x} = f(x, \alpha), \qquad x \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}^1. \tag{3.4.0}$$

$$f(p_0^0, \alpha_0) = 0 \qquad (\textit{equilibrium}), \tag{H.0.i}$$

and

$$A_0 = f_x(p_0^0, \alpha_0) \text{ has eigenvalues } \pm\, i\,\omega_0,\ \ \omega_0 > 0 \qquad (\textit{bifurcation}). \tag{H.0.ii}$$

(Equivalently, $\mathrm{tr}(A_0) = 0$ and $\det(A_0) = \omega_0^2 > 0$.)

Exercise. The implicit function theorem gives a locally defined, locally unique, smooth curve $(p^0(\alpha), \alpha)$ of equilibria through (p_0^0, α_0).

For each α, linearize at the equilibrium $p^0(\alpha)$:

$$A(\alpha) = f_x(p^0(\alpha), \alpha)$$

Assume that for α increasing through α_0, the eigenvalues $\mu(\alpha) \pm i\omega(\alpha)$ cross the imaginary axis with nonzero "speed":

$$a = \mu'(\alpha_0) = \tfrac{1}{2}\,\tfrac{d}{d\alpha}\,\mathrm{tr}(A(\alpha))\Big|_{\alpha=\alpha_0} \neq 0 \qquad (\textit{transversality}), \tag{H.1}$$

Now use the projection method and Poincaré normal form theory (modified for families) to obtain

$$\dot{\zeta}_1 = \lambda_1(\alpha)\,\zeta_1 + c_1(\alpha)\,|\zeta_1|^2\zeta_1 + O(|\zeta_1|^4), \quad \zeta_1 \in \mathbb{C},$$

or in polar coordinates $\zeta_1 = r\,e^{i\theta}$ and expanding in α

$$\dot{r} = a(\alpha - \alpha_0)\,r + \operatorname{Re} c_1(\alpha_0)\,r^3 + O(|\alpha - \alpha_0|^2\,r + |\alpha - \alpha_0|\,r^3 + r^4),$$

$$\dot{\theta} = \omega_0 + O(|\alpha - \alpha_0| + r^2),$$

If $\alpha = \alpha_0$, notice that (3.4.0) becomes Example 3.B, and we evaluate the cubic normal form coefficient b with formula $(**)$, Lecture 22. Since $(**)$ does not depend on any α-derivatives, we could set $\alpha = \alpha_0$ in (3.4.0) to calculate b (see below).

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Finally, assume

$$b = \operatorname{Re} c_1(\alpha_0) = \operatorname{Re}\left[\frac{1}{2}\,g_{21}(\alpha_0) + \frac{i}{\omega_0}\,\frac{1}{2}\,g_{20}(\alpha_0)\,g_{11}(\alpha_0)\right] \neq 0 \qquad \text{(H.2)}$$

(*nondegeneracy*),

then the family (3.4.0) has a **Hopf bifurcation**:

Theorem 3.6. *If* $f : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2$ *is* C^4 *in an open set containing* (p_0^0, α_0) *and satisfies the four conditions* (H.0.i)–(H.2), *then the family*

$$\frac{dx}{dt} = f(x, \alpha) \quad \textit{at } (p_0^0, \alpha_0)$$

has a Hopf bifurcation, locally topologically equivalent to the normal form

$$\begin{aligned} \frac{dy_1}{ds} &= a\,\beta\,y_1 - \omega_0\,y_2 + b\,(y_1^2 + y_2^2)\,y_1 \\ \frac{dy_2}{ds} &= \omega_0\,y_1 + a\,\beta\,y_2 + b\,(y_1^2 + y_2^2)\,y_2 \end{aligned} \qquad \text{at } ((0,0),0).$$

By rescaling variables and possibly changing the sign of β, we could obtain the topological normal form with $\omega_0 = 1$, $a = 1$, $b = \pm 1$.

In polar coordinates $y_1 = \rho\,\cos(\phi)$, $y_2 = \rho\,\sin(\phi)$, the normal form is

$$\begin{aligned} \frac{d\rho}{ds} &= a\beta\,\rho + b\,\rho^3, \\ \frac{d\phi}{ds} &= \omega_0, \end{aligned} \qquad (\rho,\phi) \in \mathbb{R}_+ \times \mathbb{S}^1,$$

which is used to easily determine the existence and stability of limit cycles (HW 3). One verifies there exist limit cycles bifurcating from the equilibrium at $\beta = 0$, which are stable or unstable depending on the sign of b. If $b < 0$ the Hopf bifurcation is called **supercritical**; if $b > 0$ the Hopf bifurcation is called **subcritical**.

E.g. if $a > 0$ and $b < 0$:

For reference, we summarize here the practical computations to analyze a Hopf bifurcation in a family of 2-dimensional systems (3.4.0).

1) Find the family of equilibria $p^0(\alpha) \in \mathbb{R}^2$. Linearize at $p^0(\alpha)$ and verify (H.0.i), (H.0.ii) and (H.1). Write the system in the form (3.4.1).

2) To verify (H.2): Put $A_0 = A(\alpha_0)$ and find a complex eigenvector $q_0 \neq 0 \in \mathbb{C}$,

$$A_0 \, q_0 = i \, \omega_0 \, q_0, \quad q_0 = \begin{pmatrix} q_{01} \\ q_{02} \end{pmatrix}$$

and also find the (complex) adjoint eigenvector $p_0 \in \mathbb{C}$,

$$A_0^{\intercal} \, p_0 = -i \, \omega_0 \, p_0, \quad p_0 = \begin{pmatrix} p_{01} \\ p_{02} \end{pmatrix} \in \mathbb{C}^2,$$

normalized so that

$$\langle \, p_0, q_0 \, \rangle = \bar{p}_{01} q_{01} + \bar{p}_{02} q_{02} = 1,$$

We automatically (you should check) have

$$\langle \, p_0, \bar{q}_0 \, \rangle = \bar{p}_{01} \bar{q}_{01} + \bar{p}_{02} \bar{q}_{02} = 0.$$

Any $u \in \mathbb{R}^2$ can be written uniquely as

$$u = z_1 q_0 + \bar{z}_1 \bar{q}_0, \qquad z_1 = \langle \, p_0, u \, \rangle \in \mathbb{C}$$

i.e. in components

$$u_1 = z_1 q_{01} + \bar{z}_1 \bar{q}_{01},$$

$$u_2 = z_1 q_{02} + \bar{z}_1 \bar{q}_{02}.$$

Substitute this into (3.4.1) and take the inner product with the adjoint eigenvector p_0 to obtain the equation for $\dot{z}_1$ for $\alpha = \alpha_0$,

$$\dot{z}_1 = g(z_1, \bar{z}_1, \alpha_0) = \langle\, p_0, \hat{f}(z_1 q_0 + \bar{z}_1 \bar{q}_0, \alpha_0)\,\rangle. \tag{3.4.4}$$

Expand the resulting vector field in (3.4.4) in powers of z_1 and $\bar{z}_1$ (treating z_1, $\bar{z}_1$ as *independent* variables):

where

Identify the coefficients $\frac{1}{2}g_{20}(\alpha_0)$, $g_{11}(\alpha_0)$ and $\frac{1}{2}g_{21}(\alpha_0)$, of z_1^2, $z_1\bar{z}_1$ and $z_1^2\bar{z}_1$, respectively, in this expansion. Calculate the cubic normal form coefficient b and determine its sign (if, hopefully, $b \neq 0$).

Example 3.C. Bulk oscillations in the *Brusselator* chemical reaction

$$\begin{aligned} \dot{x}_1 &= \gamma - x_1 - \alpha\, x_1 + x_1^2 x_2 \\ \dot{x}_2 &= \alpha\, x_1 - x_1^2 x_2 \end{aligned} \tag{3.C.0}$$

We suppose $\gamma > 0$ is fixed, and treat α as the bifurcation parameter. The implicit function theorem is not needed here to find the family of equilibria $p^0(\alpha)$, since the equilibria can be found explicitly (**Exercise**),

$$x_1 = p_1^0(\alpha) = \gamma, \quad x_2 = p_2^0(\alpha) = \frac{\alpha}{\gamma}.$$

A coordinate change

$$x_1 = \gamma + u_1, \quad x_2 = \frac{\alpha}{\gamma} + u_2 \tag{I}$$

transforms (3.C.0) into

or

Linearized stability:

$$\sigma(\alpha) = \mathrm{tr}(A(\alpha)) = \alpha - 1 - \gamma^2, \quad \Delta(\alpha) = \det(A(\alpha)) = \gamma^2 > 0$$

so $u = 0$ has purely imaginary eigenvalues $\pm i\sqrt{\Delta(\alpha_0)} = \pm i\,\gamma$ if and only if

$$\alpha = \alpha_0, \quad \text{where} \quad \alpha_0 = 1 + \gamma^2.$$

Thus (3.C.0) satisfies (H.0.i) and (H.0.ii) with

$$\alpha_0 = 1 + \gamma^2, \quad p_0^0 = \left(\gamma\,,\, \frac{1+\gamma^2}{\gamma}\right) \quad \omega_0 = \gamma,$$

where $\gamma > 0$, and it is easy to verify (H.1):

Now, to verify (H.2): put $\alpha = \alpha_0 = 1 + \gamma^2$ in (3.C.1),

Solve (carefully!) $A_0\, q = i\gamma\, q$, $A_0^\intercal\, p = -i\gamma\, p$, $\langle\, p, q\,\rangle = 1$:

Put $u = z_1 q + \bar{z}_1 \bar{q}$, i.e.

in (3.C.1), take the inner product with p to get

Expand in powers of z_1 and $\bar{z}_1$

identify the three important coefficients

and find the cubic normal form coefficient