MATH 552 (2023W1) Lecture 24: Fri Nov 3

[**Last lecture:** ... Hopf bifurcation ...]

Example 3.C, continued. *Brusselator* chemical reaction model

$$\begin{aligned} \dot{x}_1 &= \gamma - x_1 - \alpha\, x_1 + x_1^2 x_2 \\ \dot{x}_2 &= \alpha\, x_1 - x_1^2 x_2 \end{aligned} \tag{3.C.0}$$

We suppose $\gamma > 0$ is fixed, and treat α as the bifurcation parameter. Family of equilibria can be found explicitly (**Exercise**),

$$x_1 = p_1^0(\alpha) = \gamma, \quad x_2 = p_2^0(\alpha) = \frac{\alpha}{\gamma}.$$

The first coordinate change

$$x_1 = \gamma + u_1, \quad x_2 = \frac{\alpha}{\gamma} + u_2 \tag{I}$$

transforms (3.C.0) into

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha - 1 & \gamma^2 \\ -\alpha & -\gamma^2 \end{pmatrix}}_{A(\alpha)} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \left(\frac{\alpha}{\gamma}\, u_1^2 + 2\gamma\, u_1 u_2 + u_1^2 u_2 \right) \tag{1}$$

or

$$\dot{u} = A(\alpha)\, u + \hat{f}^{(2)}(u, \alpha) + \hat{f}^{(3)}(u, \alpha). \tag{3.C.1}$$

By linear stability analysis, it is easy to verify that (3.C.0) satisfies (H.0.i) and (H.0.ii) with

$$\alpha_0 = 1+\gamma^2, \quad p_0^0 = \left(\gamma\,,\,\frac{1+\gamma^2}{\gamma}\right) \quad \omega_0 = \gamma,$$

where $\gamma > 0$.

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It is also easy to verify (H.1):

Now, to verify (H.2): put $\alpha = \alpha_0 = 1+\gamma^2$ in (3.C.1),

Solve (carefully!) $A_0\,q = i\gamma\,q$, $A_0^{\mathsf{T}}\,p = -i\gamma\,p$, $\langle\, p, q\,\rangle = 1$:

Put $u = z_1 q + \bar{z}_1 \bar{q}$, i.e.

in (3.C.1), take the inner product with p to get

Expand in powers of z_1 and $\bar{z}_1$

identify the three important coefficients

and find the cubic normal form coefficient

By Theorem 3.6, the dynamics of (3.C.0) at (p_0^0, α_0) can be deduced from the normal form system in polar coordinates

so there are bifurcating stable limit cycles for (3.C.0) if $\alpha > 1 + \gamma^2$, at least for α sufficiently near $1 + \gamma^2$.

A schematic (Auto-style) one-parameter *local* branching diagram (showing max and min values of $x_2^0(t)$ on the limit cycle that exists for $\alpha > 1+\gamma^2$)

A two-parameter bifurcation diagram in the (α, γ)-plane with *local* phase portraits for (3.C.0), and for α *near* $1+\gamma^2$:

To study branches of limit cycles for α farther from the bifurcation value, typically need numerical computation, e.g. Auto or MatCont.

Centre manifolds

Centre manifold theory is applied, to a vector field or a map, to locally "reduce" the dynamical system to a dynamical system in a lower state space dimension. The theory is easily adapted to families of vector fields or maps. With centre manifold theory, we can analyze local bifurcations (fold, Hopf, etc.) in an n-dimensional system, even if n is large.

We summarize the theory for a vector field

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n \tag{3.5}$$

(the theory for a map is similar). At a nonhyperbolic equilibrium p^0, the $n \times n$ matrix $A = f_x(p^0)$ has a centre subspace T^c of dimension $n_0 > 0$, and there is a smooth, locally invariant **local centre manifold** $W^c_{loc}(p^0)$ with the same dimension n_0. More precisely, there is the following theorem:

Theorem 3.7. (**Local Centre Manifold**) *If* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is* C^p $(p \geq 1)$ *in an open set containing* p^0*, if* $f(p^0) = 0$*, and if* $A = f_x(p^0)$ *has* $n_0 > 0$ *eigenvalues* λ_j*, counting multiplicities, with Re* $\lambda_j = 0$*, then there exists a* C^p *submanifold* $W^c_{loc}(p^0)$ *in* $\mathbb{R}^n$*, of dimension* n_0*, that is locally invariant for* (3.5)*, contains* p^0*, and is tangent to the centre subspace at* p^0*. Moreover, there is an open neighbourhood* U *of* p^0 *in* $\mathbb{R}^n$*, such that if a solution* $x(t)$ *for* (3.5) *satisfies* $x(t) \in U$ *for all* $t \geq 0$ [*for all* $t \leq 0$]*, then* $x(t) \to W^c_{loc}(p^0)$ *as* $t \to +\infty$ [*as* $t \to -\infty$].

We develop a method for using the centre manifold theorem (the Reduction Principle) to determine dynamics restricted to a local centre manifold. Suppose the centre, stable and unstable subspaces of the linearization $A = f_x(p^0)$ have dimensions

$$\dim T^c = n_0, \quad \dim T^s = n_-, \quad \dim T^u = n_+,$$

respectively, with $0 < n_0 < n,\ 0 < n_\pm \triangleq n_- + n_+ < n,\ n_0 + n_\pm = n$. Let

$$T^{su} = T^s \oplus T^u$$

be the **stable-unstable** subspace. Then we have

$$\mathbb{R}^n = T^c \oplus T^{su}, \qquad \dim T^c = n_0, \quad \dim T^{su} = n_\pm,$$

and there exists a corresponding coordinate shift (I) and linear change of coordinates (II), from $x \in \mathbb{R}^n$, to $(u, v) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n_\pm}$, so that in the new coordinates (u, v) the equilibrium is the origin $(0, 0)$ and the linearization of the vector field at the equilibrium has a block-diagonal form (for example, real normal form),

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} B & O \\ O & C \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} g(u, v) \\ h(u, v) \end{pmatrix}, \qquad (u, v) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n_\pm}, \tag{3.6}$$

where

and both nonlinear functions

$$g : \mathbb{R}^{n_0} \times \mathbb{R}^{n_\pm} \to \mathbb{R}^{n_0}$$

$$h : \mathbb{R}^{n_0} \times \mathbb{R}^{n_\pm} \to \mathbb{R}^{n_\pm}$$

are locally defined at the origin $(u, v) = (0, 0)$, are C^p and $O(\|(u, v)\|^2)$ (if $p \geq 2$).

In these coordinates, the local centre manifold can be represented as the graph of a C^p function

where

is defined and smooth in an open neighbourhood of $u = 0$ in $\mathbb{R}^{n_0}$, and $V(u) = O(\|u\|^2)$ (i.e. $V(0) = 0$ and $V_u(0) = 0$). The dynamics restricted to $W^c_{loc}(p^0)$ essentially determine the local dynamics of the full system: