MATH 552 (2023W1) Lecture 25: Mon Nov 6

[**Last lecture:** ... 2-dim Hopf bifurcation (example).]

Centre manifolds

Centre manifold theory is applied, to a vector field or to a map, to locally "reduce" the dynamical system to a dynamical system in a lower state space dimension. The theory is easily adapted to families of vector fields or maps. With centre manifold theory, we can analyze local bifurcations (fold, Hopf, etc.) in an n-dimensional system, even if n is large.

We summarize the theory for a vector field

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n \tag{3.5}$$

(the theory for a map is similar). At a nonhyperbolic equilibrium p^0, the $n \times n$ matrix $A = f_x(p^0)$ has a centre subspace T^c of dimension $n_0 > 0$, and there is a smooth, locally invariant **local centre manifold** $W^c_{loc}(p^0)$ with the same dimension n_0. More precisely, there is the following theorem:

Theorem 3.7. (**Local Centre Manifold**) *If* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is* C^p $(p \geq 1)$ *in an open set containing* p^0*, if* $f(p^0) = 0$*, and if* $A = f_x(p^0)$ *has* $n_0 > 0$ *eigenvalues* λ_j*, counting multiplicities, with Re* $\lambda_j = 0$*, then there exists a* C^p *submanifold* $W^c_{loc}(p^0)$ *in* $\mathbb{R}^n$*, of dimension* n_0*, that is locally invariant for* (3.5)*, contains* p^0*, and is tangent to the centre subspace at* p^0*. Moreover, there is an open neighbourhood* U *of* p^0 *in* $\mathbb{R}^n$*, such that if a solution* $x(t)$ *for* (3.5) *satisfies* $x(t) \in U$ *for all* $t \geq 0$ [*for all* $t \leq 0$]*, then* $x(t) \to W^c_{loc}(p^0)$ *as* $t \to +\infty$ [*as* $t \to -\infty$].

We now develop a method (Reduction Principle) for using the centre manifold theorem (3.7) to determine dynamics restricted to a local centre manifold. Suppose the centre, stable and unstable subspaces of the linearization $A = f_x(p^0)$ have dimensions

$$\dim T^c = n_0, \quad \dim T^s = n_-, \quad \dim T^u = n_+,$$

respectively, with $0 < n_0 < n,\ 0 < n_\pm \triangleq n_- + n_+ < n,\ n_0 + n_\pm = n$. Let

$$T^{su} = T^s \oplus T^u$$

be the **stable-unstable** subspace. Then we have

$$\mathbb{R}^n = T^c \oplus T^{su}, \qquad \dim T^c = n_0, \quad \dim T^{su} = n_\pm,$$

and there exists a corresponding coordinate shift (I) and linear change of coordinates (II), from $x \in \mathbb{R}^n$, to $(u, v) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n_\pm}$, so that in the new coordinates (u, v) the equilibrium is the origin $(0, 0)$ and the linearization of the vector field at the equilibrium has a block-diagonal form (for example, real normal form),

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} B & O \\ O & C \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} g(u,v) \\ h(u,v) \end{pmatrix}, \qquad (u, v) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n_\pm}, \tag{3.6}$$

where

and both nonlinear functions

$$g : \mathbb{R}^{n_0} \times \mathbb{R}^{n_\pm} \to \mathbb{R}^{n_0}$$

$$h : \mathbb{R}^{n_0} \times \mathbb{R}^{n_\pm} \to \mathbb{R}^{n_\pm}$$

are locally defined at the origin $(u, v) = (0, 0)$, are C^p and $O(\|(u, v)\|^2)$ (if $p \geq 2$).

In these coordinates, the local centre manifold can be represented as the graph of a C^p function

where

is defined and smooth in an open neighbourhood of $u = 0$ in $\mathbb{R}^{n_0}$, and $V(u) = O(\|u\|^2)$ (i.e. $V(0) = 0$ and $V_u(0) = 0$). The dynamics restricted to $W^c_{loc}(p^0)$ essentially determine the local dynamics of the full system:

Theorem 3.8. (**Reduction Principle**) *Under the above hypotheses,* (3.6)*at* $(0,0)$ *is locally topologically equivalent to*

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} B & O \\ O & C \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} g(u, V(u)) \\ 0 \end{pmatrix} \quad at\ (0,0). \tag{3.7}$$

If there is more than one local centre manifold, then all the resulting systems (3.7) *with the different* V *are locally smoothly equivalent.*

In order to find $V(u)$, we note that local invariance of the centre manifold implies $v(t) = V(u(t))$ for all local solutions $(u(t), v(t))$ of (3.6) with $v(0) = V(u(0))$, and therefore by differentiating and using (3.6) we find that $V(u)$ must satisfy the first-order differential equation

$$C\,V(u) + h(u, V(u)) = V_u(u)\,[\,B\,u + g(u, V(u))\,]. \tag{3.8}$$

Note that the nonlinear term $h(u,v)$ in (3.6) is required for the reduction to (3.7), even though it is absent from (3.7) . We may use Taylor expansion to find an approximate solution for (3.8). Then the first component of (3.7),

$$\dot{u} = B\,u + g(u, V(u)), \qquad u \in \mathbb{R}^{n_0}$$

represents the local dynamics restricted to the centre manifold $W^c_{loc}(p^0)$ (projected onto the centre subspace). It is a good idea to consider this last equation *before* calculating any Taylor expansion coefficients of $V(u)$ explicitly, to see which specific coefficients are likely to be needed. It would be a mistake not to calculate all of the coefficients that affect the local dynamics, while it would be inefficient to calculate more coefficients than are needed.

Example 3.D. Use centre manifold theory to determine local behaviour (e.g. stability) at the origin (an equilibrium), for

$$\begin{aligned} \dot{x}_1 &= x_1^3 - x_1 x_2, \\ \dot{x}_2 &= -x_2 + 2\,x_1^2, \end{aligned} \qquad x = (x_1, x_2) \in \mathbb{R}^2.$$

Equilibrium: $(0,0)$ (no coordinate shift needed)

Linearization at equilibrium: has matrix

(already in form required for Theorem 3.8, no linear change of coordinates needed)

Find local centre manifold $W^c_{loc}((0,0))$, as $v = V(u)$ (Taylor expansion)

Local invariance: $(u(0), v(0)) \in W^c_{loc} \Rightarrow$ solution $(u(t), v(t)) \in W^c_{loc}$, at least for t belonging to some open interval that contains $t = 0$

a nonlinear DE for $V(u)$. We know $V(u) = O(|u|^2)$, expand

with coefficients v_2, v_3, etc. and get

We can solve for as many coefficients as we like, but how many will be enough?

The reduced equation will be

so if $(1 - v_2) \neq 0$ then its sign will be enough to determine the sign of $\dot{u}$, for all sufficiently small u. Start by just finding one coefficient v_2, and if $v_2 \neq 1$, that is all we need.

Find the coefficient v_2: substitute the Taylor expansion for $V(u)$ into the equation for local invariance.

At order u^2:

thus the reduced equation (representing the flow restricted to W^c_{loc}) is

By Theorem 3.8 (Reduction Principle) the original system is locally topologically equivalent at $(0, 0)$ to

The phase portrait of the original system (there is a $\mathbb{Z}_2$-equivariance under $(u, v) \mapsto (-u, v)$ and this implies the flow is reflection-symmetric about the v-axis)

(careful numerical simulation can give a more accurate picture if desired).

In this example, simply ignoring the nonlinear centre manifold (put $v = 0$ in the $\dot{u}$ equation, the so-called "tangent space approximation", which *sometimes* works but is not guaranteed to) would give the wrong stability prediction.

To analyze local bifurcations in families of n-dimensional vector fields

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \ \alpha \in \mathbb{R}^m$$

we can simply apply centre manifold theory (Theorems 3.7, 3.8 etc.) to the extended vector field

$$\begin{pmatrix} \dot{\alpha} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 \\ f(x, \alpha) \end{pmatrix}, \qquad \tilde{x} = (\alpha, x) \in \mathbb{R}^{m+n}.$$

Similarly, we can analyze local bifurcations in families of n-dimensional maps

$$x \mapsto f(x, \alpha), \quad x \in \mathbb{R}^n, \ \alpha \in \mathbb{R}^m$$

by considering the extended map

$$\begin{pmatrix} \dot{\alpha} \\ \dot{x} \end{pmatrix} \mapsto \begin{pmatrix} \alpha \\ f(x, \alpha) \end{pmatrix}, \qquad \tilde{x} = (\alpha, x) \in \mathbb{R}^{m+n}.$$