MATH 552 (2023W1) Lecture 26: Wed Nov 8

[**Last lecture:** Centre manifold theory ...]

Example 3.D, continued. Using centre manifold theory to determine local behaviour (e.g. stability) at the origin (a nonhyperboli equilibrium), for

$$\begin{aligned} \dot{x}_1 &= x_1^3 - x_1 x_2, \\ \dot{x}_2 &= -x_2 + 2\,x_1^2, \end{aligned} \qquad x = (x_1, x_2) \in \mathbb{R}^2.$$

Equilibrium $(0,0)$, nonhyperbolic (eigenvalues 0, -1).

$$u = x_1, \quad v = x_2$$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} u^3 - uv \\ 2u^2 \end{pmatrix}.$$

$$u = V(u) = v_2\,u^2 + O(|u|^3); \qquad v_2 = 2$$

By Theorem 3.8 (Reduction Principle), locally topologically equivalent at $(0,0)$ to

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} u^3 - u\,[\,2u^2 + O(|u|^3)\,] \\ 0 \end{pmatrix},$$

or

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The phase portrait of the original system (there is a $\mathbb{Z}_2$-equivariance under $(u, v) \mapsto (-u, v)$ and this implies the flow is reflection-symmetric about the v-axis)

(careful numerical simulation can give a more accurate picture if desired).

In this example, simply ignoring the nonlinear centre manifold (put $v = 0$ in the $\dot{u}$ equation, the so-called "tangent space approximation", which *sometimes* works but is not guaranteed to) would give the wrong stability prediction.

To analyze local bifurcations in families of n-dimensional vector fields

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \ \alpha \in \mathbb{R}^m$$

we can simply apply centre manifold theory (Theorems 3.7, 3.8 etc.) to the extended vector field

$$\begin{pmatrix} \dot{\alpha} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 \\ f(x, \alpha) \end{pmatrix}, \qquad \tilde{x} = (\alpha, x) \in \mathbb{R}^{m+n}.$$

Similarly, we can analyze local bifurcations in families of n-dimensional maps

$$x \mapsto f(x, \alpha), \quad x \in \mathbb{R}^n, \ \alpha \in \mathbb{R}^m$$

by considering the extended map

$$\begin{pmatrix} \dot{\alpha} \\ \dot{x} \end{pmatrix} \mapsto \begin{pmatrix} \alpha \\ f(x, \alpha) \end{pmatrix}, \qquad \tilde{x} = (\alpha, x) \in \mathbb{R}^{m+n}.$$

Example 3.E. Local bifurcation analysis for the family

$$\begin{aligned} \dot{x}_1 &= \alpha + \alpha(x_1 + 2x_2) - x_1^2 + x_1 x_2 - 2x_2^2, \\ \dot{x}_2 &= -2x_2 + x_1^2 \end{aligned} \qquad (x_1, x_2) \in \mathbb{R}^2, \ \alpha \in \mathbb{R}^1$$

Notice that for $\alpha = \alpha_0 = 0$, the point $(x_1, x_2) = p_0^0 = (0, 0)$ is an equilibrium, and it is nonhyperbolic (eigenvalues 0, -2).

Write the family in $\mathbb{R}^2$ as an extended vector field in $\mathbb{R}^3$

with equilibrium $(\alpha, x_1, x_2) = (0, 0, 0)$. The eigenvalues of the 3-dimensional system at the origin are clearly 0, 0, -2, i.e. 0 is an eigenvalue of multiplicity 2. Moreover, the linearization at the equilibrium is already in the block-diagonal form that appears in Theorem 3.8.

In the extended system at the origin we have

$$\text{centre subspace} = \tilde{T}^c = (\alpha, x_1)\text{-plane} = (u_1, u_2)\text{-plane},$$

$$\text{stable-unstable subspace} = \text{stable subspace} = \tilde{T}^{su} = x_2\text{-axis} = v\text{-axis}$$

with dimensions

$$\dim(\tilde{T}^c) = 2, \quad \dim(\tilde{T}^{su}) = 1$$

In the notation of Theorem 3.8 we let

$$g(u,v) = \begin{pmatrix} 0 \\ u_1u_2 + 2u_1v - u_2^2 + u_2v - 2v^2 \end{pmatrix} \in \mathbb{R}^2,$$

$$h(u,v) = u_2^2 \in \mathbb{R}^1,$$

and the local centre manifold $\tilde{W}^c_{loc}(0,0,0)$ is expressed as

But before finding coefficients explicitly, look ahead to see which coefficients are likely to be needed: the reduced equation will be and the second component (the first component is trival) is

Now review Theorem 3.1 (the fold bifurcation theorem). By Theorem 3.1, none of the higher order terms affect the qualitative dynamics; the family is locally topologically equivalent to

So in this example, *none* of the Taylor expansion coefficients of $v = V(u)$ are needed explicitly, and the extended system in $\mathbb{R}^3$ is locally topologically equivalent to

$$\dot{u}_1 = 0$$

$$\dot{u}_2 = u_1 - u_2^2$$

$$\dot{v} = -2v$$

which is easy enough to analyze explicitly, the 3-dimensional phase portrait looks like

and taking 2-dimensional slices $\{u_1 = \text{constant}\}$ of the 3-dimensional phase portrait we get

The actual local centre manifold for the original extended system would typically have *some* curvature, but the phase portraits above are correct, *up to* local topological equivalence. For example, typical actual phase portraits for the original system might be something like

The projection method and local bifurcations in n dimensions

Consider the family

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \ \alpha \in \mathbb{R}^1 \tag{3.9}$$

and assume

$$f(0, \alpha_0) = 0 \qquad\qquad (equilibrium), \tag{F.0.i'}$$

(a coordinate shift may already have been applied so that the origin is the equilibrium in these coordinates), and $A_0 = f_x(0, \alpha_0)$ is nonhyperbolic . To apply Theorem 3.8, it is not always efficient to block-diagonalize A_0 if n is large and $n_0 = \dim T^c$ is small. Instead, a projection method can be more efficient (and can be generalized to infinite-dimensional state spaces).

Example: the fold bifurcation for vector fields in n dimensions.

Suppose $n \geq 2$, and, in addition to (F.0.i'), assume

$$A_0 = f_x(0, \alpha_0) \text{ has a simple eigenvalue 0 and all other eigenvalues}$$
$$\text{have nonzero real parts} \qquad (\textit{bifurcation}). \tag{F.0.ii'}$$

We have the analogues, in n dimensions, of (F.0.i) and (F.0.ii) from Theorem 3.1, and we expect that generically we would have a fold bifurcation. To verify there is indeed a fold bifurcation, we use the projection method.

Let $q \in \mathbb{R}^n$ be an eigenvector corresponding to the simple eigenvalue 0

then for $\alpha = \alpha_0$ the centre subspace is

thus every vector in T^c has the form $u\,q \in \mathbb{R}^n$, for some $u \in \mathbb{R}^1$. To find this scalar u easily, we find the normalized adjoint eigenvector $p \in \mathbb{R}^n$

Then we can use p to project any vector x onto T^c: it can be proved (using the Fredholm Alternative Theorem, Appendix B, p.6) that any $x \in \mathbb{R}^n$ can be uniquely decomposed as

$$x = u\,q + y, \qquad u \in \mathbb{R}^1, \quad y \in \mathbb{R}^n, \tag{3.10}$$

with

and

Write (3.9) at $\alpha = \alpha_0$ in the form

$$\dot{x} = A_0\, x + \underbrace{\tfrac{1}{2} B_0\, [x, x]}_{f^{(2)}(x,\alpha_0)} + O(\|x\|^3) \tag{3.11}$$

where

$$B_0\, [\,\cdot\,,\,\cdot\,] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

is a symmetric **bilinear** function or map (linear in each of the two slots), the i-th component of $B_0\, [v, w] \in \mathbb{R}^n$ is

$$B_{0,i}\, [v, w] = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 f_i}{\partial x_j \partial x_k}(0, \alpha_0)\, v_j\, w_k, \quad \text{for } v, w \in \mathbb{R}^n.$$

Substituting (3.10) into (3.11) and taking the inner product with the adjoint eigenvector p, we get essentially the u-component of (3.7) in the reduced system of Theorem 3.8