MATH 552 (2023W1) Lecture 27: Fri Nov 10

[**Last lecture:** ... centre manifold theory ...]

The projection method and local bifurcations in n dimensions

Example: the fold bifurcation for vector fields in n dimensions, continued.

Consider the family

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \ \alpha \in \mathbb{R}^1 \tag{3.9}$$

with state space dimension $n \geq 2$, and assume

$$f(0, \alpha_0) = 0 \qquad\qquad (\textit{equilibrium}), \tag{F.0.i'}$$

and

$$A_0 = f_x(0, \alpha_0) \text{ has a simple eigenvalue 0 and all other eigenvalues}$$
$$\text{have nonzero real parts} \qquad\qquad (\textit{bifurcation}). \tag{F.0.ii'}$$

Let $q \in \mathbb{R}^n$ be an eigenvector corresponding to the simple eigenvalue 0

$$A_0\, q = 0, \quad q \neq 0,$$

then for $\alpha = \alpha_0$ the centre subspace is

$$T^c = \text{span}\{q\}$$

thus every vector $x(t)$ in T^c for any t has the form

$$u(t)\, q, \quad u \in \mathbb{R}^1.$$

To find this scalar $u(t)$ easily, we find the normalized adjoint eigenvector $p \in \mathbb{R}^n$

$$A_0^\intercal p = 0, \quad \langle p, q \rangle = 1.$$

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Then we can use p to project any vector $x(t) \in \mathbb{R}^n$ onto T^c: it can be proved (using the Fredholm Alternative Theorem, Appendix B, p.6) that any $x(t) \in \mathbb{R}^n$ can be uniquely decomposed as

$$x(t) = u(t)\, q + y(t), \qquad u(t) \in \mathbb{R}^1, \quad y(t) \in \mathbb{R}^n, \tag{3.10}$$

with

and

Write (3.9) at $\alpha = \alpha_0$ in the form

$$\dot{x} = A_0\, x + \underbrace{\tfrac{1}{2} B_0\, [x, x]}_{f^{(2)}(x, \alpha_0)} + O(\|x\|^3) \tag{3.11}$$

where

$$B_0\, [\,\cdot\,,\,\cdot\,] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

is a symmetric **bilinear** function or map (linear in each of the two slots), the i-th component of $B_0[v,w] \in \mathbb{R}^n$ is

$$B_{0,i}[v,w] = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 f_i}{\partial x_j \partial x_k}(0, \alpha_0)\, v_j\, w_k, \quad \text{for } v, w \in \mathbb{R}^n.$$

Substituting (3.10) into (3.11) and taking the inner product with the adjoint eigenvector p, we get essentially the u-component of (3.7) in the reduced system of Theorem 3.8

Since $y \in T^{su}$, the representation of the local centre manifold as a graph in this setup is

and so we have

$$\dot{u} = \underbrace{\langle\, p, \tfrac{1}{2} B_0\,[q, q]\,\rangle}_{b}\; u^2 + O(|u|^3), \qquad u \in \mathbb{R}^1$$

which represents (3.9) with $\alpha = \alpha_0$, restricted to the local centre manifold.

In n dimensions, the conditions (F.1) and (F.2) for a fold bifurcation in Theorem 3.1 are replaced by

$$a = \langle\, p, f_\alpha(0, \alpha_0)\,\rangle \neq 0 \qquad (\textit{transversality}), \tag{F.1'}$$

$$b = \langle\, p, \tfrac{1}{2} B_0\,[q, q]\,\rangle \neq 0 \qquad (\textit{nondegeneracy}). \tag{F.2'}$$

Exercise. Analyze Example 3.E using the projection method and the four conditions (F.0.i')–(F.2'), to verify there is a fold bifurcation.

The Hopf bifurcation in $\mathbb{R}^n$ can be treated similarly, but it's somewhat more complicated, because in the calculation of the normal form coefficient at order 3, the Taylor expansion of the centre manifold function $y = V(z_1, \bar{z}_1) = O(\|(z_1, \bar{z}_1)\|^2)$ interacts with the Poincaré normal form calculations (see pp. 177–180 in the textbook).

4. Topics in Global Dynamics and Bifurcations

Homoclinic orbits

Suppose a vector field

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n.$$

has an equilibrium $p^0 \in \mathbb{R}^n$. An orbit $\Gamma = \{x^0(t)\}_{t\in\mathbb{R}}$ is called **homoclinic** to p^0 if $\Gamma \neq \{p^0\}$, $\lim_{t\to\infty} x^0(t) = p^0$ and $\lim_{t\to-\infty} x^0(t) = p^0$.

A homoclinic orbit to a hyperbolic equilibrium is *structurally unstable*: if there is a vector field with such a homoclinic orbit, then there exist arbitrarily small perturbations of the vector field such that the perturbed vector field in some open neighbourhood U, of $\Gamma \cup \{p^0\}$, is not topologically equivalent to the original vector field, in particular, the perturbed vector field has *no* homoclinic orbit.

The homoclinic bifurcation for 2-dimensional vector fields

Homoclinic orbits are often associated with other dynamics of interest ("homoclinic phenomena"). As an introductory example, we summarize a bifurcation analysis near a homoclinic orbit Γ for a family of vector fields in $\mathbb{R}^2$. For the unperturbed system, this orbit is homoclinic to a hyperbolic saddle point p_0^0. The homoclinic orbit together with the equilibrium, $\Gamma \cup \{ p_0^0 \}$, is sometimes called a "saddle loop", or "separatrix loop", or "separatrix cycle".

Consider a 1-parameter family of 2-dimensional vector fields

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^2, \ \alpha \in \mathbb{R}^1. \tag{4.1}$$

Suppose there exist $p_0^0 \in \mathbb{R}^2$, $\alpha_0 \in \mathbb{R}^1$ such that

$$f(p_0^0, \alpha_0) = 0 \qquad (\textit{equilibrium}), \tag{SL.0.i}$$

$$A_0 = f_x(p_0^0, \alpha_0) \text{ has eigenvalues } \lambda_{10} < 0 < \lambda_{20}$$
$$(\textit{hyperbolic saddle}), \tag{SL.0.ii}$$

$$\dot{x} = f(x, \alpha_0) \text{ has an orbit } \Gamma = \{x^0(t)\} \text{ that is homoclinic to } p_0^0$$
$$(\textit{bifurcation}). \tag{SL.0.iii}$$

Thus for $\alpha = \alpha_0$, (4.1) has a hyperbolic saddle equilibrium p_0^0 and there is a homoclinic orbit $\Gamma = \{x^0(t)\}$, with $\lim_{t\to\pm\infty} x^0(t) = p_0^0$. Generically, for $\alpha \neq \alpha_0$, α near α_0, there is no homoclinic orbit for (4.1) in some open neighbourhood U (fixed in $\mathbb{R}^2$, for different α) of $\Gamma \cup \{ p_0^0 \}$. We would like to know if anything of interest happens for (4.1), in U, if $\alpha \neq \alpha_0$.

By the now familiar arguments (e.g. HW 3), the implicit function theorem can be used to solve $f(x, \alpha) = 0$, to obtain a unique, locally defined, smooth solution

giving a smooth curve $(p^0(\alpha), \alpha)$ of isolated equilibria through the point (p_0^0, α_0) in $\mathbb{R}^2 \times \mathbb{R}^1$.

And, since the eigenvalues of A_0 are simple, the matrix of the linearization $A(\alpha) = f_x(p^0(\alpha), \alpha)$ has real eigenvalues $\lambda_1(\alpha)$, $\lambda_2(\alpha)$ that depend smoothly on α near α_0, with $\lambda_j(\alpha_0) = \lambda_{j0}$, $j = 1, 2$, and thus by continuity

and $p^0(\alpha)$ remains a hyperbolic saddle equilibrium, for all α sufficiently near α_0. So, locally (i.e. in a sufficiently small open neighbourhood of p_0^0), "nothing much happens" when α is perturbed from α_0. Globally, it might be more interesting.

As mentioned above, condition (SL.0.iii) is structurally unstable: generically the stable and unstable manifolds of $p^0(\alpha)$ will "split" for $\alpha \neq \alpha_0$.

The usual coordinate shift

$$x = p^0(\alpha) + u \tag{I}$$

transforms (4.1) into

$$\dot{u} = f(p^0(\alpha) + u, \alpha) = \hat{f}(u, \alpha) \tag{4.2}$$

where now $\hat{f}(0, \alpha) = 0$, i.e. $u = 0$ is always the equilibrium, for all α near α_0.

Then a linear coordinate change

$$u = T(\alpha)\, v \tag{II}$$

transforms (4.2) into

$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = \tag{4.3}$$

so that the linearization is in real normal form (i.e. diagonal) for all α near α_0 , and the nonlinear terms are

In these v-coordinates, the stable subspace T^s is the v_1-axis and the unstable subspace T^u is the v_2-axis (for all α near α_0), and there exist local stable and unstable manifolds $W^s_{loc}(0, \alpha)$, $W^u_{loc}(0, \alpha)$ tangent at the origin to T^s, T^u respectively.

Then, there exists (see the textbook p. 203 for a proof) a *global* change of coordinates (a diffeomorphism, a "local linearization")

$$v = H(y, \alpha) \tag{III}$$

that transforms (4.3) into

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \tag{4.4}$$

where now in the y-coordinates, the local stable and unstable manifolds actually *coincide* with intervals of the new coordinate axes near the origin, for all α sufficiently close to α_0

with

for all (y_2, α) near $(0, \alpha_0)$,

for all (y_1, α) near $(0, \alpha_0)$.