MATH 552 (2023W1) Lecture 28: Fri Nov 17

[**Last lecture:** ... centre manifold theory (fold bifurcation in n dimensions by projection method). Homoclinic orbits. 2-d homoclinic (saddle-loop, Andronov-Leontovich) bifurcation ...

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Consider a 1-parameter family of 2-dimensional vector fields

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^2, \ \alpha \in \mathbb{R}^1. \tag{4.1}$$

Suppose there exist $p_0^0 \in \mathbb{R}^2$, $\alpha_0 \in \mathbb{R}^1$ such that

$$f(p_0^0, \alpha_0) = 0 \qquad (\textit{equilibrium}), \tag{SL.0.i}$$

$$A_0 = f_x(p_0^0, \alpha_0) \text{ has eigenvalues } \lambda_{10} < 0 < \lambda_{20}$$

$$(\textit{hyperbolic saddle}), \tag{SL.0.ii}$$

$$\dot{x} = f(x, \alpha_0) \text{ has an orbit } \Gamma = \{x^0(t)\} \text{ that is homoclinic to } p_0^0$$

$$(\textit{bifurcation}). \tag{SL.0.iii}$$

The implicit function theorem can be used to solve $f(x, \alpha) = 0$, to obtain a unique, locally defined, smooth solution

$$x = p^0(\alpha),$$

giving a smooth curve $(p^0(\alpha), \alpha)$ of isolated equilibria through the point (p_0^0, α_0) in $\mathbb{R}^2 \times \mathbb{R}^1$.

And, since the eigenvalues of A_0 are simple, the matrix of the linearization $A(\alpha) = f_x(p^0(\alpha), \alpha)$ has real eigenvalues $\lambda_1(\alpha)$, $\lambda_2(\alpha)$ that depend smoothly on α near α_0, with $\lambda_j(\alpha_0) = \lambda_{j0}$, $j = 1, 2$, and thus by continuity

$$\lambda_1(\alpha) < 0 < \lambda_2(\alpha)$$

and $p^0(\alpha)$ remains a hyperbolic saddle equilibrium, for all α sufficiently near α_0. As mentioned above, condition (SL.0.iii) is structurally unstable: generically the stable and unstable manifolds of $p^0(\alpha)$ will "split" for $\alpha \neq \alpha_0$.

After three coordinate changes, we arrive at

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$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \tag{4.4}$$

where now in the new y-coordinates, the local stable and unstable manifolds *coincide* with intervals of the new coordinate axes near the origin, for all α sufficiently close to α_0

with

for all (y_2, α) near $(0, \alpha_0)$,

for all (y_1, α) near $(0, \alpha_0)$.

Now, for some fixed $\varepsilon > 0$ sufficiently small, we define a cross-section Σ for (4.4)

and a family of Poincaré maps

We study the flow that produces the Poincaré map, in two stages.

Stage I: $y(t)$ near the saddle 0 in $\mathbb{R}^2$, for α near α_0.

Define another cross-section

and a family of ("near-to-saddle") maps

for $0 < \xi_0 < \varepsilon$.

For all sufficiently small $\varepsilon > 0$, the solution of (4.4) with initial condition in Σ^+ at $t = 0$

$$y_1(0) = \varepsilon, \quad y_2(0) = \xi_0$$

is approximated (near the hyperbolic equilibrium) by the linear flow

Using this linear flow, the orbit starting in Σ^+ first reaches the cross-section Π at the unique positive time $t = t^*$ that solves $y_2(t) = \varepsilon$, and we obtain an explicit (but approximate) expression for the map Δ:

Stage II: "Near-to-homoclinic" global map, $y(t)$ near Γ in $\mathbb{R}^2$, for α near α_0. Define another family of maps

for $-\varepsilon < \eta < \varepsilon$ (notice, η is allowed to be zero or negative).

$\eta = 0$ corresponds to the initial value $(y_1(0), y_2(0)) = (0, \varepsilon) \in W^u(0, \alpha)$.

Let

$$\beta(\alpha) = Q(0, \alpha)$$

For all α near α_0, $\beta(\alpha)$ is a **split function** ("A-L version") that measures the *signed* distance, measured along the *oriented* cross-section Σ, *from* $W^s(0, \alpha)$ (branch 1) *to* $W^u(0, \alpha)$ (branch 1). Thus, for $\alpha = \alpha_0$, assuming the existence of a homoclinic orbit in (SL.0.iii) is equivalent to assuming

$$(\textit{bifurcation}). \quad \text{(SL.0.iii')}$$

and the Taylor expansion of $\beta(\alpha)$ about α_0 is

Assume a generic condition holds,

$$a = \beta'(\alpha_0) \neq 0. \qquad (\textit{transversality}) \qquad \text{(SL.1)}$$

From the theory of ODEs, we know $Q(\eta, \alpha)$ is as smooth as the family of vector fields that generates it, so assuming (4.1) is smooth, we can make a Taylor expansion of Q, about $\eta = 0$ for any fixed α

then expand the α-dependent coefficients about $\alpha = \alpha_0$

What can we know about the Taylor coefficient $Q_\eta(0, \alpha_0)$?

For $\alpha = \alpha_0$ in particular, we know $Q(\,\cdot\,, \alpha_0) : \mathbb{R}^1 \to \mathbb{R}^1$ must be a local diffeomorphism, smoothly invertible, and thus at $\eta = 0$

Furthermore, ODE theory says that orbits cannot cross and therefore

Thus we know $Q(\eta, \alpha)$ must have a Taylor expansion of the form

with $Q_\eta(0, \alpha_0) > 0$.

The family of Poincaré maps. Now we compose $P = Q \circ \Delta$, i.e.

Assume another generic condition holds

or equivalently,

$$\sigma_0 = \lambda_{10} + \lambda_{20} = \mathrm{div}(f(p_0^0, \alpha_0)) \neq 0 \qquad (\textit{nondegeneracy}) \qquad \text{(SL.2)}$$

where $\mathrm{div}(f) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = \mathrm{tr}(f_x)$, which is practical to verify in the original family (4.1).

It has been proved, under our five assumptions, that for all sufficiently small $\varepsilon > 0$, the approximations we made do in fact determine the dynamics of the family of Poincaré maps.

Theorem 4.1. (**Andronov & Leontovich**) *If* $f : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2$ *is* C^2 *and satisfies the five conditions* (SL.0.i)–(SL.2)*, then* (4.1) *has a family of Poincaré maps that is locally topologically equivalent to*

$$\xi \mapsto \beta + b\,\xi^{-\lambda_{10}/\lambda_{20}}, \quad \xi > 0, \qquad \textit{at } (0,0),$$

where b *is a positive constant. In particular, for all* α *sufficiently near* α_0 *there is an open neighbourhood* U *of* $\Gamma \cup \{p_0^0\}$ *in* $\mathbb{R}^2$*, in which a unique limit cycle* L_β *for* (4.1) *bifurcates from* $\Gamma \cup \{p_0^0\}$ *for* α *on only one side of* α_0*. If* $\sigma_0 < 0$*, then* L_β *exists only for* $\beta > 0$ *and is stable, while if* $\sigma_0 > 0$*, then* L_β *exists only for* $\beta < 0$ *and is unstable.*

Notice that the "saddle quantity" σ_0 in (SL.2) is computed directly from the original vector field (4.1) with $\alpha = \alpha_0$.

The defined split function $\beta = \beta(\alpha) = a\,(\alpha - \alpha_0) + O(|\alpha - \alpha_0|^2)$ changes sign as α increases past α_0. The sign of the coefficient a in (SL.1) can be deduced from numerical experiments, by observing how the homoclinic orbit Γ splits, as α increases past α_0. There is also an analytic way to verify (SL.1), see the textbook p. 211.

Exercise. Work out what happens in the case $\sigma_0 > 0$.

If $\alpha \to \alpha_0$ from the appropriate side of α_0, then three things happen: (i) $\beta \to 0$, (ii) the limit cycle L_β approaches the separatrix cycle $\Gamma \cup \{p_0^0\}$, and (iii) the period of L_β approaches infinity.