MATH 552 (2023W1) Lecture 29: Mon Nov 20

[**Last lecture:** ... two-dimensional homoclinic (saddle-loop, Andronov-Leontovich) bifurcation ...]

A 1-parameter family of 2-dimensional vector fields

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^2, \ \alpha \in \mathbb{R}^1. \tag{4.1}$$

with

$$f(p_0^0, \alpha_0) = 0 \qquad (\textit{equilibrium}), \tag{SL.0.i}$$

$$A_0 = f_x(p_0^0, \alpha_0) \text{ has eigenvalues } \lambda_{10} < 0 < \lambda_{20}$$
$$(\textit{hyperbolic saddle}), \tag{SL.0.ii}$$

$$\dot{x} = f(x, \alpha_0) \text{ has an orbit } \Gamma = \{x^0(t)\} \text{ that is homoclinic to } p_0^0$$
$$(\textit{bifurcation}). \tag{SL.0.iii}$$

The implicit function theorem, coordinate changes, 2-stage Poincaré map. 1st stage, near the hyperbolic saddle equilibrium, approximate the flow with linear flow to get an approximate map

$$\Delta(\xi_0, \alpha) \approx \varepsilon^{1+(\lambda_{10}/\lambda_{20})} \, \xi_0^{-\lambda_{10}/\lambda_{20}}.$$

So far, in the analysis of the 2nd stage, near the homoclinic orbit,

$$Q(\,\cdot\,, \alpha) : \Pi \to \Sigma, \quad (\eta, \varepsilon) \mapsto (\varepsilon, \xi_1), \quad \xi_1 = Q(\eta, \alpha)$$

for $-\varepsilon < \eta < \varepsilon$ (notice, η is allowed to be zero or negative). $\eta = 0$ corresponds to the initial value $(y_1(0), y_2(0)) = (0, \varepsilon) \in W^u(0, \alpha)$:

$$\beta(\alpha) = Q(0, \alpha)$$

For all α near α_0, $\beta(\alpha)$ is a **split function** ("A-L version") that measures the *signed* distance, measured along the *oriented* cross-section Σ, *from* $W^s(0, \alpha)$ (branch 1) *to* $W^u(0, \alpha)$ (branch 1), and the Taylor expansion of $\beta(\alpha)$ about α_0 is

$$\beta(\alpha) = a \cdot (\alpha - \alpha_0) + O(|\alpha - \alpha_0|^2),$$

where we assume a generic condition holds,

$$a = \beta'(\alpha_0) \neq 0. \qquad (\textit{transversality}) \qquad \text{(SL.1)}$$

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From the theory of ODEs, we know $Q(\eta, \alpha)$ is as smooth as the family of vector fields that generates it, so assuming (4.1) is smooth, we can make a Taylor expansion of Q, about $\eta = 0$ for any fixed α

then expand the α-dependent coefficients about $\alpha = \alpha_0$

What can we know about the Taylor coefficient $Q_\eta(0, \alpha_0)$?

For $\alpha = \alpha_0$ in particular, we know $Q(\,\cdot\,, \alpha_0) : \mathbb{R}^1 \to \mathbb{R}^1$ must be a local diffeomorphism, smoothly invertible, and thus at $\eta = 0$

Furthermore, ODE theory says that orbits cannot cross

and therefore

Thus we know $Q(\eta, \alpha)$ must have a Taylor expansion of the form

with $Q_\eta(0, \alpha_0) > 0$.

The family of Poincaré maps. Now we compose $P = Q \circ \Delta$, i.e.

Assume another generic condition holds

or equivalently,

$$\sigma_0 = \lambda_{10} + \lambda_{20} = \operatorname{div}(f(p_0^0, \alpha_0)) \neq 0 \qquad (\textit{nondegeneracy}) \qquad \text{(SL.2)}$$

where $\operatorname{div}(f) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = \operatorname{tr}(f_x)$, which is practical to verify in the original family (4.1).

It has been proved, under our five assumptions, that for all sufficiently small $\varepsilon > 0$, the approximations we made do in fact determine the dynamics of the family of Poincaré maps.

Theorem 4.1. (**Andronov & Leontovich**) *If* $f : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2$ *is* C^2 *and satisfies the five conditions* (SL.0.i)–(SL.2)*, then* (4.1) *has a family of Poincaré maps that is locally topologically equivalent to*

$$\xi \mapsto \beta + b\,\xi^{-\lambda_{10}/\lambda_{20}}, \quad \xi > 0, \qquad \textit{at } (0,0),$$

where b is a positive constant. In particular, for all α sufficiently near α_0 there is an open neighbourhood U of $\Gamma \cup \{p_0^0\}$ in $\mathbb{R}^2$, in which a unique limit cycle L_β for (4.1) *bifurcates from $\Gamma \cup \{p_0^0\}$ for α on only one side of α_0. If $\sigma_0 < 0$, then L_β exists only for $\beta > 0$ and is stable, while if $\sigma_0 > 0$, then L_β exists only for $\beta < 0$ and is unstable.*

Notice that the "saddle quantity" σ_0 in (SL.2) is computed directly from the original vector field (4.1) with $\alpha = \alpha_0$.

The defined split function $\beta = \beta(\alpha) = a\,(\alpha - \alpha_0) + O(|\alpha - \alpha_0|^2)$ changes sign as α increases past α_0. The sign of the coefficient a in (SL.1) can be deduced from numerical experiments, by observing how the homoclinic orbit Γ splits, as α increases past α_0. There is also an analytic way to verify (SL.1), see the textbook p. 211.

Exercise. Work out what happens in the case $\sigma_0 > 0$.

If $\alpha \to \alpha_0$ from the appropriate side of α_0, then three things happen: (i) $\beta \to 0$, (ii) the limit cycle L_β approaches the separatrix cycle $\Gamma \cup \{p_0^0\}$, and (iii) the period of L_β approaches infinity.

Melnikov's method

Melnikov's method is a global perturbation method for detecting homoclinic solutions. In its most basic form, we perturb from a 2-dimensional Hamiltonian vector field that has a homoclinic solution, to analytically determine if a nearby non-Hamiltonian system has a homoclinic solution (or not). The basic method has been generalized to a variety of settings.

We start with a 2-dimensional Hamiltonian vector field (the "unperturbed" system) that is generated by a Hamiltonian function $H : \mathbb{R}^2 \to \mathbb{R}$,

$$\dot{x} = f_0(x), \qquad x \in \mathbb{R}^2, \tag{4.5}$$

where

$$f_0(x) = \begin{pmatrix} f_{10}(x_1, x_2) \\ f_{20}(x_1, x_2) \end{pmatrix}, \qquad f_{10} = \frac{\partial H}{\partial x_2}, \quad f_{20} = -\frac{\partial H}{\partial x_1},$$

and we assume

$$\begin{aligned} &\text{(4.5) has a hyperbolic saddle equilibrium } p_0^0, \\ &\text{and an orbit } \Gamma = \{x^0(t)\} \text{ homoclinic to } p_0^0, \\ &\lim_{t \to \pm\infty} x^0(t) = p_0^0. \end{aligned} \tag{M.0}$$

Now we consider a family of "perturbed" 2-dimensional systems of *nonautonomous, periodic* ODEs

$$\dot{x} = f(t, x, \alpha) = f_0(x) + \alpha f_1(x, \omega t), \quad t \in \mathbb{R},\ x \in \mathbb{R}^2,\ \alpha \in \mathbb{R}^1, \quad (4.6)$$

where $\omega = 2\pi/T_0 > 0$ is fixed, α is a parameter near 0, and f is periodic in t with period $T_0 > 0$

$$f(t + T_0, x, \alpha) = f(t, x, \alpha) \quad \text{for all } t,\ x,\ \alpha,$$

and so f_1 is periodic in ωt with period 2π

$$f_1(x, \omega t + 2\pi) = f_1(x, \omega t) \quad \text{for all } t,\ x.$$

Denote the unique solution $x(t)$ of (4.6) that satisfies the initial condition

$$x(t_0) = x_0 \in \mathbb{R}^2,$$

by

$$x(t) = \varphi(t, t_0, x_0, \alpha) \in \mathbb{R}^2.$$

Recall from the basic existence-uniqueness-smoothness Theorem 2.1, that $\varphi(t, t_0, x_0, \alpha)$ is smooth in all its variables. We will have occasion to study the derivative with respect to the parameter $\Theta(t) = \varphi_\alpha(t, t_0, x_0, \alpha)$ (see HW 2 problem 1(b)).

Defining a new variable $\theta = \omega t$, we write the family of 2-dimensional nonautonomous differential equations (4.6) as a family of 3-dimensional *autonomous* differential equations, or vector fields (see Examples 2.D–E),

$$\begin{aligned} \dot{x} &= \\ \dot{\theta} &= \end{aligned} \tag{4.7}$$

Define a global cross-section for (4.7) for all α (near 0)

and we study the family of 2-dimensional Poincaré maps for (4.7),

The unique solution of (4.7) that satisfies the initial condition

$$\tilde{x}(0) = (\, x(0)\,,\, \theta(0)\,) = (\, x_0\,,\, 0\,(\mathrm{mod}\ 2\pi)\,) \in \Sigma_0$$

is

$$\tilde{x}(t) = (\, \varphi(t, 0, x_0, \alpha)\,,\, \omega t\,(\mathrm{mod}\ 2\pi)\,) \in X$$

and the natural coordinate representation of the Poincaré map is

For the unperturbed $\alpha = 0$ system, the Poincaré map $P(\,\cdot\,, 0)$ just happens to be the time-$(2\pi/\omega)$ (stroboscopic) map for the 2-dimensional flow $\varphi(t, 0, x_0, 0)$, of the unperturbed autonomous Hamiltonian system (4.5), therefore $P(\,\cdot\,, 0)$ has a hyperbolic saddle fixed point $(p_0^0, 0 \,(\mathrm{mod}\, 2\pi)) \in \Sigma_0$, whose 1-dimensional stable and unstable manifolds intersect, and happen to coincide along the smooth curve $\Gamma_0 \times \{\, 0 \,(\mathrm{mod}\ 2\pi)\,\}$ in Σ_0.

Then, taking points in Σ_0 as initial values at $t = 0$ for $(4.7)_{\alpha=0}$, the hyperbolic saddle fixed point $(p_0^0, 0\,(\mathrm{mod}\ 2\pi))$ for the 2-dimensional Poincaré map $P(\,\cdot\,,0)$ in Σ_0 generates a $(2\pi/\omega)$-periodic hyperbolic limit cycle $\tilde{L}_0 = \{\,\tilde{p}(t,0) = (p_0^0, \omega t\,(\mathrm{mod}\,2\pi))\,\}_{t\in\mathbb{R}}$ for the 3-dimensional vector field $(4.7)_{\alpha=0}$ in X, and the 2-dimensional (global) stable and unstable manifolds $\tilde{W}_0^s$ and $\tilde{W}_0^u$ of $\tilde{L}_0$ intersect, and happen to coincide along a smooth 2-dimensional *homoclinic manifold* $\tilde{\Gamma}_0 = \Gamma_0 \times \mathbb{S}^1$ in X.