MATH 552 (2023W1) Lecture 30: Wed Nov 22

[**Last lecture:** ... two-dimensional homoclinic (saddle-loop, Andronov-Leontovich) bifurcation]

A 1-parameter family of 2-dimensional vector fields

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^2, \ \alpha \in \mathbb{R}^1. \tag{4.1}$$

with

$$f(p_0^0, \alpha_0) = 0 \qquad (\textit{equilibrium}), \tag{SL.0.i}$$

$$A_0 = f_x(p_0^0, \alpha_0) \text{ has eigenvalues } \lambda_{10} < 0 < \lambda_{20}$$

$$(\textit{hyperbolic saddle}), \tag{SL.0.ii}$$

$$\dot{x} = f(x, \alpha_0) \text{ has an orbit } \Gamma = \{x^0(t)\} \text{ that is homoclinic to } p_0^0$$

$$(\textit{bifurcation}). \tag{SL.0.iii}$$

For all α near α_0, $\beta(\alpha)$ is a **split function** ("A-L version") that measures the *signed* distance, measured along the *oriented* cross-section Σ, *from* $W^s(0, \alpha)$ (appropriate branch) *to* $W^u(0, \alpha)$ (appropriate branch), and we assume a generic condition holds,

$$a = \beta'(\alpha_0) \neq 0. \qquad (\textit{transversality}) \tag{SL.1}$$

Finally, assume the generic condition

$$\sigma_0 = \lambda_{10} + \lambda_{20} = \operatorname{div}(f(p_0^0, \alpha_0)) \neq 0 \qquad (\textit{nondegeneracy}) \quad (\mathrm{SL.2})$$

where $\operatorname{div}(f) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = \operatorname{tr}(f_x)$.

Theorem 4.1. (**Andronov & Leontovich**) *If* $f : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2$ *is* C^2 *and satisfies the five conditions* (SL.0.i)–(SL.2), *then* (4.1) *has a family of Poincaré maps that is locally topologically equivalent to*

$$\xi \mapsto \beta + b\,\xi^{-\lambda_{10}/\lambda_{20}}, \quad \xi > 0, \qquad \textit{at } (0,0),$$

where b *is a positive constant. In particular, for all* α *sufficiently near* α_0 *there is an open neighbourhood* U *of* $\Gamma \cup \{p_0^0\}$ *in* $\mathbb{R}^2$, *in which a unique limit cycle* L_β *for* (4.1) *bifurcates from* $\Gamma \cup \{p_0^0\}$ *for* α *on only one side of* α_0. *If* $\sigma_0 < 0$, *then* L_β *exists only for* $\beta > 0$ *and is stable, while if* $\sigma_0 > 0$, *then* L_β *exists only for* $\beta < 0$ *and is unstable.*

Exercise. Work out what happens in the case $\sigma_0 > 0$.

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If $\alpha \to \alpha_0$ from the appropriate side of α_0, then three things happen:

(i) $\beta \to 0$, (ii) the limit cycle L_β approaches the separatrix cycle $\Gamma \cup \{p_0^0\}$, and (iii) the period of L_β approaches infinity.

Melnikov's method

Melnikov's method is a global perturbation method for detecting homoclinic solutions. In its most basic form, we perturb from a 2-dimensional Hamiltonian vector field that has a homoclinic solution, to analytically determine if a nearby non-Hamiltonian system has a homoclinic solution (or not). The basic method has been generalized to a variety of settings.

We start with a 2-dimensional Hamiltonian vector field (the "unperturbed" system) that is generated by a Hamiltonian function $H : \mathbb{R}^2 \to \mathbb{R}$,

$$\dot{x} = f_0(x), \qquad x \in \mathbb{R}^2, \tag{4.5}$$

where

$$f_0(x) = \begin{pmatrix} f_{10}(x_1, x_2) \\ f_{20}(x_1, x_2) \end{pmatrix}, \qquad f_{10} = \frac{\partial H}{\partial x_2}, \quad f_{20} = -\frac{\partial H}{\partial x_1},$$

and we assume

$$\begin{aligned} &\text{(4.5) has a hyperbolic saddle equilibrium } p_0^0, \\ &\text{and an orbit } \Gamma = \{x^0(t)\} \text{ homoclinic to } p_0^0, \\ &\lim_{t\to\pm\infty} x^0(t) = p_0^0. \end{aligned} \tag{M.0}$$

Now we consider a family of "perturbed" 2-dimensional systems of *nonautonomous, periodic* ODEs

$$\dot{x} = f(t, x, \alpha) = f_0(x) + \alpha f_1(x, \omega t), \quad t \in \mathbb{R},\, x \in \mathbb{R}^2,\, \alpha \in \mathbb{R}^1, \quad (4.6)$$

where $\omega = 2\pi/T_0 > 0$ is fixed, α is a parameter near 0, and f is periodic in t with period $T_0 > 0$

$$f(t + T_0, x, \alpha) = f(t, x, \alpha) \quad \text{for all } t,\, x,\, \alpha,$$

and so f_1 is periodic in ωt with period 2π

$$f_1(x, \omega t + 2\pi) = f_1(x, \omega t) \quad \text{for all } t,\, x.$$

Denote the unique solution $x(t)$ of (4.6) that satisfies the initial condition

$$x(t_0) = x_0 \in \mathbb{R}^2,$$

by

$$x(t) = \varphi(t, t_0, x_0, \alpha) \in \mathbb{R}^2.$$

Recall from the basic existence-uniqueness-smoothness Theorem 2.1, that $\varphi(t, t_0, x_0, \alpha)$ is smooth in all its variables. We will have occasion to study the derivative with respect to the parameter $\Theta(t) = \varphi_\alpha(t, t_0, x_0, \alpha)$ (see HW 2 problem 1(b)).

Defining a new variable $\theta = \omega t$, we write the family of 2-dimensional nonautonomous differential equations (4.6) as a family of 3-dimensional *autonomous* differential equations, or vector fields (see Examples 2.D–E),

$$\begin{aligned} \dot{x} &= \\ \dot{\theta} &= \end{aligned} \tag{4.7}$$

Define a global cross-section for (4.7) for all α (near 0)

and we study the family of 2-dimensional Poincaré maps for (4.7),

The unique solution of (4.7) that satisfies the initial condition

$$\tilde{x}(0) = (\, x(0)\, ,\, \theta(0)\,) = (\, x_0\, ,\, 0\, (\mathrm{mod}\ 2\pi)\,) \in \Sigma_0$$

is

$$\tilde{x}(t) = (\, \varphi(t, 0, x_0, \alpha)\, ,\, \omega t\, (\mathrm{mod}\ 2\pi)\,) \in X$$

and the natural coordinate representation of the Poincaré map is

For the unperturbed $\alpha = 0$ system, the Poincaré map $P(\,\cdot\,, 0)$ just happens to be the time-$(2\pi/\omega)$ (stroboscopic) map for the 2-dimensional flow $\varphi(t, 0, x_0, 0)$, of the unperturbed autonomous Hamiltonian system (4.5), therefore $P(\,\cdot\,, 0)$ has a hyperbolic saddle fixed point $(p_0^0, 0 \pmod{2\pi}) \in \Sigma_0$, whose 1-dimensional stable and unstable manifolds intersect, and happen to coincide along the smooth curve $\Gamma_0 \times \{\, 0 \pmod{2\pi} \,\}$ in Σ_0.

Then, taking points in Σ_0 as initial values at $t = 0$ for $(4.7)_{\alpha=0}$, the hyperbolic saddle fixed point $(p_0^0, 0 \,(\mathrm{mod}\ 2\pi))$ for the 2-dimensional Poincaré map $P(\,\cdot\,, 0)$ in Σ_0 generates a $(2\pi/\omega)$-periodic hyperbolic limit cycle $\tilde{L}_0 = \{\, \tilde{p}(t, 0) = (p_0^0, \omega t \,(\mathrm{mod}\, 2\pi))\,\}_{t\in\mathbb{R}}$ for the 3-dimensional vector field $(4.7)_{\alpha=0}$ in X, and the 2-dimensional (global) stable and unstable manifolds $\tilde{W}_0^s$ and $\tilde{W}_0^u$ of $\tilde{L}_0$ intersect, and happen to coincide along a smooth 2-dimensional *homoclinic manifold* $\tilde{\Gamma}_0 = \Gamma_0 \times \mathbb{S}^1$ in X.

For the perturbed system, for all $\alpha \neq 0$ sufficiently near 0, the hyperbolic saddle fixed point $(p_0^0, 0\,(\text{mod} 2\pi)) \in \Sigma_0$ for the 2-dimensional Poincaré map $P(\cdot, 0)$ persists as a fixed point $(p^0(\alpha), 0\,(\text{mod} 2\pi)) \in \Sigma_0$ for the perturbed Poincaré map $P(\cdot, \alpha)$ with $p^0(\alpha) = p_0^0 + O(|\alpha|)$, and it remains a saddle point for $P(\cdot, \alpha)$ with 1-dimensional stable and unstable subspaces. Furthermore, in Σ_0, the 1-dimensional local stable and unstable manifolds $W^s_{0,loc}$ and $W^u_{0,loc}$, of $(p_0^0, 0\,(\text{mod} 2\pi))$ for the 2-dimensional Poincaré map $P(\cdot, 0)$, persist as 1-dimensional local stable and unstable manifolds $W^s_{\alpha,loc}$ and $W^u_{\alpha,loc}$, of $(p^0(\alpha), 0\,(\text{mod} 2\pi))$ for $P(\cdot, \alpha)$, and $W^s_{\alpha,loc}$ and $W^u_{\alpha,loc}$ are $O(|\alpha|)$-close to their $\alpha = 0$ counterparts (e.g. Example 2.D, implicit function theorem, continuity of multipliers with respect to the parameter).

Now taking points in the 2-dimensional cross-section Σ_0 as initial values at $t = 0$ for the 3-dimensional vector field (4.7), it follows that the hyperbolic limit cycle $\tilde{L}_0$, of saddle type, for persists in X, as a hyperbolic limit cycle $\tilde{L}_\alpha = \{\tilde{p}^0(t,\alpha) = (p^0(t,\alpha), \omega t \,(\mathrm{mod}\, 2\pi))\}$, also of saddle type, with $p^0(t,\alpha) = p_0^0 + O(|\alpha|)$. It also follows that the 2-dimensional *local* stable and unstable manifolds $\tilde{W}^s_{0,loc}$ and $\tilde{W}^u_{0,loc}$, for $\tilde{L}_0$, also persist in X, as local stable and unstable manifolds $\tilde{W}^s_{\alpha,loc}$ and $\tilde{W}^u_{\alpha,loc}$, for $\tilde{L}_\alpha$, and these invariant manifolds also remain $O(|\alpha|)$-close to their $\alpha = 0$ counterparts $\tilde{W}^s_{0,loc}$ and $\tilde{W}^u_{0,loc}$.

We define the 2-dimensional vector field

$$f_0^{\perp}(x) = \begin{pmatrix} -f_{20}(x_1, x_2) \\ f_{10}(x_1, x_2) \end{pmatrix},$$

which is orthogonal to the unperturbed 2-dimensional (Hamiltonian) vector field f_0, and oriented.

Define a **Melnikov integral**

$$\begin{aligned} M_\alpha(\theta, 0) &= \int_{-\infty}^{+\infty} \langle\, f_0^{\perp}(x^0(t)), f_1(x^0(t), \omega t + \theta)\,\rangle\, dt \\ &= \int_{-\infty}^{+\infty} \langle\, \eta(t), f_\alpha(t + \theta/\omega, x^0(t), 0)\,\rangle\, dt, \end{aligned} \tag{4.8}$$

where $\langle\, \cdot\,,\, \cdot\,\rangle$ denotes the usual inner product in $\mathbb{R}^2$, and $\eta(t) = f_0^{\perp}(x^0(t))$.

Theorem 4.2. (**Melnikov's Method – Periodic Perturbations**) *If, in* (4.7), $f_0 : \mathbb{R}^2 \to \mathbb{R}^2$ *and* $f_1 : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2$ *are* C^2, *with* $f_1(x, \cdot)$ *periodic in its last variable with period* 2π, *and the two conditions* (M.0) *and*

$$M_\alpha(\theta_0, 0) = 0, \qquad M_{\alpha\theta}(\theta_0, 0) \neq 0 \quad \textit{for some } \theta_0 \in \mathbb{S}^1, \tag{M.1}$$

are true, then there is an open neighbourhood $\tilde{U}$ *of* $\tilde{\Gamma}_0 \cup \tilde{L}_0$ *in* X*, such that for all* $\alpha \neq 0$ *sufficiently near* 0*, the stable and unstable manifolds* $\tilde{W}^s_\alpha$ *and* $\tilde{W}^u_\alpha$ *for* (4.7) *have transversal intersections in* $\tilde{U}$*. Furthermore, if* $M_\alpha(\theta, 0) \neq 0$ *for all* $\theta \in \mathbb{S}^1$*, then for all* $\alpha \neq 0$ *sufficiently near* 0*,* $\tilde{W}^s_\alpha$ *and* $\tilde{W}^u_\alpha$ *do not intersect in* $\tilde{U}$*.*