MATH 552 (2023W1) Lecture 31: Fri Nov 24

[**Last lecture:** Melnikov's method ...]

The "unperturbed" system: 2-dimensional Hamiltonian vector field

$$\dot{x} = f_0(x), \qquad x \in \mathbb{R}^2, \tag{4.5}$$

$$f_0(x) = \begin{pmatrix} f_{10}(x_1, x_2) \\ \\ f_{20}(x_1, x_2) \end{pmatrix}, \qquad f_{10} = \frac{\partial H}{\partial x_2}, \quad f_{20} = -\frac{\partial H}{\partial x_1},$$

and we assume

$$\begin{aligned} &\text{(4.5) has a hyperbolic saddle equilibrium } p_0^0, \\ &\text{and an orbit } \Gamma_0 = \{x^0(t)\} \text{ homoclinic to } p_0^0, \\ &\lim_{t\to\pm\infty} x^0(t) = p_0^0. \end{aligned} \tag{M.0}$$

"Perturbed" 2-dimensional systems of *nonautonomous, periodic* ODEs

$$\dot{x} = f(t, x, \alpha) = f_0(x) + \alpha f_1(x, \omega t), \quad t \in \mathbb{R},\, x \in \mathbb{R}^2,\, \alpha \in \mathbb{R}^1, \tag{4.6}$$

expressed as a 3-dimensional autonomous sytem

$$\begin{aligned} \dot{x} &= f_0(x) + \alpha f_1(x, \theta), \\ \dot{\theta} &= \omega, \end{aligned} \qquad \tilde{x} = (x, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1 = X, \quad \alpha \in \mathbb{R}^1. \tag{4.7}$$

Fixed, global cross-section for (4.7) for *all* α (near 0)

$$\Sigma_0 = \{\tilde{x} = (x, \theta) \in X : \ x \in \mathbb{R}^2,\ \theta = 0 \ (\mathrm{mod}\ 2\pi)\}$$

and family of 2-dimensional Poincaré maps for (4.7),

$$P(\,\cdot\,,\alpha) : \Sigma_0 \to \Sigma_0.$$

The coordinate representation of the Poincaré map is

$$P(x_0, \alpha) = \varphi(2\pi/\omega, 0, x_0, \alpha) \in \mathbb{R}^2, \quad x_0 \in \mathbb{R}^2.$$

where $\varphi(t, 0, x_0, \alpha)$ is the unique solution of the pertubed nonautonoumous periodic system (4.6) that satisfies the initial condition

$$x(0) = \varphi(0, 0, x_0, \alpha) = x_0.$$

The perturbed ($\alpha \neq 0$) Poincaré map $P(\cdot, \alpha)$ has a hyperbolic saddle fixed point; the perturbed 3-dimensional system (4.7) has a corresponding hyperbolic saddle-type limit cycle $\tilde{L}_\alpha$. *Local* stable and unstable manifolds, for $P(\cdot, \alpha)$ and for (4.7) exist, and are $O(|\alpha)|)$-close to their unperturbed ($\alpha = 0$) counterparts.

Define the 2-dimensional vector field

$$f_0^\perp(x) = \begin{pmatrix} -f_{20}(x_1, x_2) \\ f_{10}(x_1, x_2) \end{pmatrix},$$

which is orthogonal to the unperturbed 2-dimensional (Hamiltonian) vector field f_0, and *oriented in a specific way.*

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Define the **Melnikov integral**

$$
\begin{aligned}
M_\alpha(\theta,0) &= \int_{-\infty}^{+\infty} \langle\, f_0^\perp(x^0(t)), f_1(x^0(t), \omega t + \theta)\,\rangle \, dt \\
&= \int_{-\infty}^{+\infty} \langle\, \eta(t), f_\alpha(t+\theta/\omega, x^0(t), 0)\,\rangle \, dt,
\end{aligned}
\tag{4.8}
$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the usual inner product in $\mathbb{R}^2$, and $\eta(t) = f_0^\perp(x^0(t))$.

Theorem 4.2. (**Melnikov's Method – Periodic Perturbations**) *If, in* (4.7), $f_0 : \mathbb{R}^2 \to \mathbb{R}^2$ *and* $f_1 : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2$ *are* C^2, *with* $f_1(x,\cdot)$ *periodic in its last variable with period* 2π, *and the two conditions* (M.0) *and*

$$
M_\alpha(\theta_0, 0) = 0, \qquad M_{\alpha\theta}(\theta_0, 0) \neq 0 \quad \textit{for some } \theta_0 \in \mathbb{S}^1, \tag{M.1}
$$

are true, then there is an open neighbourhood $\tilde{U}$ *of* $\tilde{\Gamma}_0 \cup \tilde{L}_0$ *in* X, *such that for all* $\alpha \neq 0$ *sufficiently near* 0, *the stable and unstable manifolds* $\tilde{W}^s_\alpha$ *and* $\tilde{W}^u_\alpha$ *for* (4.7) *have transversal intersections in* $\tilde{U}$. *Furthermore, if* $M_\alpha(\theta,0) \neq 0$ *for all* $\theta \in \mathbb{S}^1$, *then for all* $\alpha \neq 0$ *sufficiently near* 0, $\tilde{W}^s_\alpha$ *and* $\tilde{W}^u_\alpha$ *do not intersect in* $\tilde{U}$.

We have a similar theorem if the perturbed 2-dimensional system is *autonomous* and depends on another parameter $\gamma \in \mathbb{R}^1$ (a constant),

$$\dot{x} = f_0(x) + \alpha f_1(x, \gamma), \qquad x \in \mathbb{R}^2, \quad \gamma \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1, \tag{4.9}$$

In this case we can consider an extended 3-dimensional system

$$\begin{aligned} \dot{x} &= f_0(x) + \alpha f_1(x, \gamma), \\ \dot{\gamma} &= 0, \end{aligned}$$

and adapt the proof of Theorem 4.2. Now, the **Melnikov integral** is

$$M_\alpha(\gamma, 0) = \int_{-\infty}^{+\infty} \langle\, f_0^{\perp}(x^0(t)), f_1(x^0(t), \gamma) \,\rangle \, dt.$$

Theorem 4.3. (**Melnikov's Method – Autonomous Perturbations**) *If, in* (4.9), $f_0 : \mathbb{R}^2 \to \mathbb{R}^2$ *and* $f_1 : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2$ *are* C^2 *and satisfy both conditions* (M.0) *and*

$$M_\alpha(\gamma_0, 0) = 0, \qquad M_{\alpha\gamma}(\gamma_0, 0) \neq 0 \quad \textit{for some } \gamma_0 \in \mathbb{R}^1, \tag{M.2}$$

then there is an open neighbourhood U *of* $\Gamma_0 \cup \{\, p_0^0 \,\}$ *in* $\mathbb{R}^2$*, such that for all* $\alpha \neq 0$ *sufficiently near* 0*, there is a unique value* $\hat{\gamma}(\alpha) = \gamma_0 + O(|\alpha|)$ *so that* (4.9) *has a homoclinic orbit in* U *only for* $\gamma = \hat{\gamma}(\alpha)$ *and no homoclinic orbit in* U *for* $\gamma \neq \hat{\gamma}(\alpha)$*. Furthermore, if* $M_\alpha(\gamma, 0) \neq 0$ *for all* γ*, then for all* $\alpha \neq 0$ *sufficiently near* 0*,* (4.9) *has no homoclinic orbit in* U*.*

Example 4.A. A periodically forced, damped nonlinear oscillator

$$\ddot{x} + \alpha\,\dot{x} - x + x^3 = \varepsilon\,\cos(\omega t), \quad x \in \mathbb{R}^1.$$

Assume damping is positive and small, $0 < \alpha \ll 1$, and the forcing amplitude scales like α,

$$\varepsilon = \alpha\gamma, \quad \text{where } \gamma \neq 0 \text{ is fixed as } \alpha \to 0+$$

Then rewrite the nonautonomous 2nd order equation as

in the form of (4.7), with

We observe (Example 2.F, lectures 11–12) that f_0 is a Hamiltonian vector field, with Hamiltonian function

To get an analytic expression for homoclinic orbits, note that they lie on the level set

so solving for x_2 gives

etc., and the right homoclinic orbit ($x_1 > 0$) is

and

The Melnikov integral is

so if $|\,\gamma\sqrt{2}\,\pi\omega\,\mathrm{sech}\left(\frac{\pi\omega}{2}\right)| > \frac{4}{3}$, i.e. $|\gamma| > \frac{4}{3\sqrt{2}\,\pi\omega\,\mathrm{sech}(\pi\omega/2)}$

we see there exists some $\theta_0 \in \mathbb{S}^1$ satisfying (M.1), and by Theorem 4.2, $\tilde{W}^s_\alpha$ and $\tilde{W}^u_\alpha$ have transversal intersections in some open neighbourhood $\tilde{U}$ of $\tilde{\Gamma} \cup \tilde{L}_0$, for all $\alpha \neq 0$ sufficiently close to 0.

Furthermore, if $|\gamma| < \frac{4}{3\sqrt{2}\,\pi\omega\,\mathrm{sech}(\pi\omega/2)}$, then $\tilde{W}^s_\alpha$ and $\tilde{W}^u_\alpha$ have no intersections in $\tilde{U}$.

Derivation of the Melnikov integral in Theorem 4.2

Recall

$$f_0^\perp = \begin{pmatrix} -f_{20} \\ f_{10} \end{pmatrix}$$

The "vertical" cross-section:

Let

and let L be the 1-dimensional line segment of length 2ε in $\mathbb{R}^2$ oriented in the direction of η_0, for some small, fixed $\varepsilon > 0$

Observe that for $x \in L$,

$$\beta = \left\langle \frac{\eta_0}{\|\eta_0\|} \, , \, x - x_0^0 \right\rangle = \frac{\langle \, \eta_0 \, , x - x_0^0 \, \rangle}{\|\eta_0\|}$$

is the *signed* distance *from* x_0^0 *to* x, along the oriented line segment L.

Now for (4.7) in the 3-dimensional manifold X, define the 2-dimensional "vertical" cross-section Π

Keeping Π fixed in X for all α, we will study how certain solutions of (4.7) intersect Π for $\alpha = 0$ and for $\alpha \neq 0$.

Transversal intersections with the "vertical" cross-section:

For $\alpha = 0$, the stable and unstable manifolds of the "unperturbed" limit cycle, $\tilde{W}_0^s$ and $\tilde{W}_0^u$, both have transversal (in fact, orthogonal) intersections with Π. The transversal intersections form two smooth curves in Π, and in fact, the curves coincide in a single "vertical" line

$$(x_0^0,\, \theta \pmod{2\pi}) \in \Pi, \quad \theta \in \mathbb{S}^1.$$

Transversal intersections of manifolds persist under smooth changes of the manifolds, therefore for $\alpha \neq 0$, for all α sufficiently near 0, the stable manifold of the "perturbed" limit cycle, $\tilde{W}^s_\alpha$, still has a transversal intersection with Π, forming some smooth curve

$$(s(\theta,\alpha),\, \theta \ (\mathrm{mod}\ 2\pi)) \in \Pi, \quad \theta \in \mathbb{S}^1,$$

with

$$s(\theta,\alpha) = x_0^0 + O(|\alpha|) \in L \quad \text{for all } \theta \in \mathbb{S}^1.$$

Similarly, $\tilde{W}^u_\alpha$ still has a transversal intersection with Π, forming some smooth curve

$$(u(\theta,\alpha),\, \theta \ (\mathrm{mod}\ 2\pi)) \in \Pi, \quad \theta \in \mathbb{S}^1,$$

with

$$u(\theta,\alpha) = x_0^0 + O(|\alpha|) \in L \quad \text{for all } \theta \in \mathbb{S}^1.$$

The Melnikov function:

A **split function** defined as

$$\beta(\theta,\alpha) = \frac{\langle\, \eta_0\,, u(\theta,\alpha) - s(\theta,\alpha)\,\rangle}{\|\eta_0\|}$$

measures the *signed* "horizontal" distance (i.e. at $\theta =$ constant) *from* $\tilde{W}^s_\alpha$ *to* $\tilde{W}^u_\alpha$, along Π.

The **Melnikov function** is defined to be the numerator of $\beta(\theta,\alpha)$

$$M(\theta,\alpha) = \langle\, \eta_0\,,\ u(\theta,\alpha) - s(\theta,\alpha)\,\rangle = \|\eta_0\|\,\beta(\eta,\alpha)$$

which is simply a positive constant times the split function. It is a smooth function, so it has a Taylor expansion in α, at $\alpha = 0$.

For $\alpha = 0$, $\tilde{W}_0^s$ and $\tilde{W}_0^u$ coincide along Π, and this implies

$$M(\theta, 0) = 0 \quad \text{for all } \theta \in \mathbb{S}^1$$

and therefore we can "factor out" α from the Taylor expansion of $M(\theta, \alpha)$ about $\alpha = 0$