MATH 552 (2023W1) Lecture 32: Mon Nov 27

[**Last lecture:** ... Melnikov's method (derivation) ...]

The "unperturbed" system: 2-dimensional Hamiltonian vector field

$$\dot{x} = f_0(x), \qquad x \in \mathbb{R}^2, \tag{4.5}$$

$$f_0(x) = \begin{pmatrix} f_{10}(x_1, x_2) \\ \\ f_{20}(x_1, x_2) \end{pmatrix}, \qquad f_{10} = \frac{\partial H}{\partial x_2}, \quad f_{20} = -\frac{\partial H}{\partial x_1},$$

and we assume

$$\begin{aligned} &\text{(4.5) has a hyperbolic saddle equilibrium } p_0^0, \\ &\text{and an orbit } \Gamma_0 = \{x^0(t)\} \text{ homoclinic to } p_0^0, \\ &\lim_{t\to\pm\infty} x^0(t) = p_0^0. \end{aligned} \tag{M.0}$$

"Perturbed" 2-dimensional system of *nonautonomous, periodic* ODEs

$$\dot{x} = f_0(x) + \alpha f_1(x, \omega t), \quad x \in \mathbb{R}^2,\ \alpha \in \mathbb{R}^1, \tag{4.6}$$

expressed as an equivalent, 3-dimensional autonomous sytem

$$\begin{aligned} \dot{x} &= f_0(x) + \alpha f_1(x, \theta), \\ \dot{\theta} &= \omega, \end{aligned} \qquad \tilde{x} = (x, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1,\ \alpha \in \mathbb{R}^1. \tag{4.7}$$

Fixed, "vertical" cross-section for (4.7) for *all* α (near 0)

$$\Pi = L \times \mathbb{S}^1 = \{(\tilde{x} = (x, \theta) \in X : \ x \in L,\ \theta \in \mathbb{S}^1\}$$

where

$$L: \;\; x = x_0^0 + \beta\,\frac{\eta_0}{\|\eta_0\|}, \quad |\beta| < \varepsilon,$$

$$\beta = \frac{\langle\, \eta_0 \,,\, x - x_0^0 \,\rangle}{\|\eta_0\|}, \quad \eta_0 = f_0^{\perp}(x_0^0) \in \mathbb{R}^2, \quad x_0^0 = x^0(0).$$

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Transversal intersections with the "vertical" cross-section:

For $\alpha = 0$, the stable and unstable manifolds of the "unperturbed" limit cycle, $\tilde{W}_0^s$ and $\tilde{W}_0^u$, both have transversal (in fact, orthogonal) intersections with Π. The transversal intersections form two smooth curves in Π, and in fact, these two curves coincide in a single "vertical" line

$$(x_0^0,\, \theta \pmod{2\pi}) \in \Pi, \quad \theta \in \mathbb{S}^1.$$

Transversal intersections of manifolds persist under smooth changes of the manifolds, therefore for $\alpha \neq 0$, for all α sufficiently near 0, $\tilde{W}_\alpha^s$ still has a transversal intersection with Π, forming some smooth curve

$$(s(\theta, \alpha),\, \theta \pmod{2\pi}) \in \Pi, \quad \theta \in \mathbb{S}^1,$$

with

$$s(\theta, \alpha) = x_0^0 + O(|\alpha|) \in L \quad \text{for all } \theta \in \mathbb{S}^1.$$

Similarly, $\tilde{W}_\alpha^u$ still has a transversal intersection with Π, forming some smooth curve

$$(u(\theta, \alpha),\ \theta \ (\mathrm{mod}\ 2\pi)) \in \Pi, \quad \theta \in \mathbb{S}^1,$$

with

$$u(\theta, \alpha) = x_0^0 + O(|\alpha|) \in L \quad \text{for all } \theta \in \mathbb{S}^1.$$

The Melnikov function:

The **split function** ("M version") defined as

$$\beta(\theta, \alpha) = \frac{\langle\, \eta_0\,, u(\theta, \alpha) - s(\theta, \alpha)\,\rangle}{\|\eta_0\|}$$

measures the *signed* "horizontal" distance (i.e. at $\theta =$ constant) *from* $\tilde{W}^s_\alpha$ *to* $\tilde{W}^u_\alpha$, along Π.

The **Melnikov function**, defined to be the numerator of $\beta(\theta, \alpha)$,

$$M(\theta, \alpha) = \langle\, \eta_0\,,\ u(\theta, \alpha) - s(\theta, \alpha)\,\rangle = \|\eta_0\|\, \beta(\eta, \alpha),$$

is simply a positive constant times the split function. It is a smooth function, so it has a Taylor expansion in α, at $\alpha = 0$.

For $\alpha = 0$, $\tilde{W}^s_0$ and $\tilde{W}^u_0$ coincide along Π, and this implies

$$M(\theta, 0) = 0 \quad \text{for all } \theta \in \mathbb{S}^1$$

and therefore we can "factor out" α from the Taylor expansion of $M(\theta, \alpha)$ about $\alpha = 0$

Clearly, for all $\alpha \neq 0$ sufficiently near 0,

$$\beta(\theta, \alpha) = 0 \quad \Leftrightarrow \quad M(\theta, \alpha) = 0 \quad \Leftrightarrow \quad \tilde{M}(\theta, \alpha) = 0.$$

Notice we can solve

$$\tilde{M}(\theta, \alpha) = 0$$

exactly, for θ as an explicit function of α, using the implicit function theorem, if there exists some $\theta_0 \in \mathbb{S}^1$ such that

i.e.

which is just the condition (M.1).

We can find an analytic expression for the leading-order term $M_\alpha(\theta, 0)$, of $\tilde{M}(\theta, \alpha)$, by studying the derivative of solutions to initial value problems with respect to the parameter α, evaluated at $\alpha = 0$.

Derivation of the integral formula (4.8):

Fix some $\theta_0 \in \mathbb{S}^1$, and consider two different initial value problems for the 3-dimensional system (4.7), with two initial conditions in Π,

$$\tilde{x}^u(0) = (\, u(\theta_0, \alpha),\ \theta_0 \,(\mathrm{mod}\ 2\pi)\,), \quad \tilde{x}^s(0) = (\, s(\theta_0, \alpha),\ \theta_0 \,(\mathrm{mod}\ 2\pi)\,).$$

The solutions of (4.7) with these initial conditions will have the form

$$\tilde{x}^u(t) = (\, x^u(t, \theta_0, \alpha),\ \omega t + \theta_0 \,(\mathrm{mod}\ 2\pi)\,),$$

$$\tilde{x}^s(t) = (\, x^s(t, \theta_0, \alpha),\ \omega t + \theta_0 \,(\mathrm{mod}\ 2\pi)\,),$$

respectively, with $x^u(0, \theta_0, \alpha) = u(\theta_0, \alpha)$, and $x^s(0, \theta_0, \alpha) = s(\theta_0, \alpha)$.

The first two components of $\tilde{x}^u(t)$ and $\tilde{x}^s(t)$ can be expressed in terms of solutions of initial value problems for the nonautonomous 2-dimensional system (4.6). Note that we have to start the corresponding solutions at the correct phase of the forcing function, $f_1(x, \theta_0)$:

Now we define two "time-dependent Melnikov functions" (whose purpose will become clear later)

$$M^u(t, \theta_0, \alpha) = \langle\, f_0^{\perp}(x^0(t)),\, x^u(t, \theta_0, \alpha)\,\rangle, \quad -\infty < t \leq 0,$$

$$M^s(t, \theta_0, \alpha) = \langle\, f_0^{\perp}(x^0(t)),\, x^s(t, \theta_0, \alpha)\,\rangle, \quad 0 \leq t < +\infty.$$

When $t = 0$, we have

We now find an integral expression for $M^u_\alpha(0, \theta_0, 0)$. Take the α-derivative of $M^u(t, \theta_0, \alpha)$ at $\alpha = 0$

and then take the t-derivative at any $t < 0$

recalling that

Exercise: Use the results of HW 2 problem 1 to show that

$$\psi(t) = x_\alpha^u(t, \theta_0, 0)$$

satisfies the linear nonhomogeneous ODE

Using the result of this Exercise, we get

Now we integrate, to recover $M^u_\alpha(0, \theta_0, 0)$

By the fundamental theorem of calculus,

so we have

We calculate

and we have

$$M^u_\alpha(0,\theta_0,0) = \int_{-\infty}^{0} \langle\, f_0^\perp(x^0(t)),\; f_1(x^0(t),\omega t+\theta_0)\,\rangle\, dt.$$

Similarly (**Exercise**),

$$M^s_\alpha(0,\theta_0,0) = -\int_{0}^{\infty} \langle\, f_0^\perp(x^0(t)),\; f_1(x^0(t),\omega t+\theta_0)\,\rangle\, dt$$

and finally

which is the Melnikov integral (4.8).

Generalizations of Theorems 4.2 and 4.3.

Melnikov's method can be generalized to work in situations where the unperturbed system is not Hamiltonian. For example, a useful generalization can be made in the setting of Theorem 4.1 (the Andronov-Leontovich theorem on the homoclinic bifurcation in $\mathbb{R}^2$): recall we have a 1-parameter family of 2-dimensional vector fields (autonomous ODEs)

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^2, \ \alpha \in \mathbb{R}^1. \tag{4.1}$$

and the unperturbed ($\alpha = \alpha_0$) vector field satisfies

$$f(p_0^0, \alpha_0) = 0 \qquad\qquad (\textit{equilibrium}), \tag{SL.0.i}$$

$$A_0 = f_x(p_0^0, \alpha_0) \text{ has eigenvalues } \lambda_{10} < 0 < \lambda_{20}$$

$$(\textit{hyperbolic saddle}), \tag{SL.0.ii}$$

$$\dot{x} = f(x, \alpha_0) \text{ has an orbit } \Gamma = \{x^0(t)\} \text{ that is homoclinic to } p_0^0$$

$$(\textit{bifurcation}). \tag{SL.0.iii}$$

In Theorem 4.1, if the unperturbed vector field $f(\,\cdot\,, \alpha_0)$ is not Hamiltonian (if it is Hamiltonian, then (SL.2) *cannot* be satisfied), consider a Melnikov integral

$$M_\alpha(\alpha_0) = \int_{-\infty}^{+\infty} \langle\, \eta(t), f_\alpha(x^0(t), \alpha_0)\,\rangle \, dt,$$

where (see the textbook, p. 211)

$$\eta(t) = \exp\left[-\int_0^t \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)\Big|_{(x,\alpha)=(x^0(\tau),\alpha_0)} d\tau\right] f^\perp(x^0(t), \alpha_0),$$

and the condition on the derivative of the ("A-L") split function (SL.1) is equivalent to an analytic expression (for the "M" split function)

$$M_\alpha(\alpha_0) \neq 0. \tag{SL.1'}$$

Also, the vector function $\eta(t)$ can be characterized as the unique-up-to-a-scalar-multiple bounded solution of the "adjoint variational equation"

$$\dot{\eta} = -A(t)^\intercal \eta, \qquad \eta \in \mathbb{R}^2, \quad t \in \mathbb{R}$$

(i.e. the adjoint variational equation has a 1-dimensional function space of bounded solutions, $\operatorname{span}\{\eta(t)\}$), where $A(t) = f_x(x^0(t), \alpha_0)$ is the linearization of the unperturbed vector field at the homoclinic solution.

For perturbations of systems with homoclinic orbits in dimensions $n \geq 3$, and also for perturbations of systems with heteroclinic solutions, generalized Melnikov integrals are defined in terms of an inner product with an $\eta(t) \in \mathbb{R}^n$, a unique-up-to-a-scalar-multiple bounded solution of the adjoint variational equation. This approach generalizes to infinite dimensions (PDEs, DDEs), and is useful, e.g. for studying stability and bifurcation of travelling waves in reaction-diffusion systems.