MATH 552 (2023W1) Lecture 33: Wed Nov 29

[**Last lecture:** ... Melnikov's method (derivation) ...]

The "unperturbed" system: 2-dimensional Hamiltonian vector field

$$\dot{x} = f_0(x), \qquad x \in \mathbb{R}^2, \tag{4.5}$$

$$f_0 = \begin{pmatrix} f_{10} \\ f_{20} \end{pmatrix}, \qquad f_{10} = \frac{\partial H}{\partial x_2}, \quad f_{20} = -\frac{\partial H}{\partial x_1},$$

and we assume

$$\begin{aligned} &\text{(4.5) has a hyperbolic saddle equilibrium } p_0^0, \\ &\text{and an orbit } \Gamma_0 = \{x^0(t)\} \text{ homoclinic to } p_0^0, \\ &\lim_{t\to\pm\infty} x^0(t) = p_0^0. \end{aligned} \tag{M.0}$$

"Perturbed" 2-dimensional system of *nonautonomous, periodic* ODEs

$$\dot{x} = f_0(x) + \alpha f_1(x, \omega t), \quad x \in \mathbb{R}^2,\ \alpha \in \mathbb{R}^1, \tag{4.6}$$

expressed as an equivalent, 3-dimensional autonomous sytem

$$\begin{aligned} \dot{x} &= f_0(x) + \alpha f_1(x, \theta), \\ \dot{\theta} &= \omega, \end{aligned} \qquad \tilde{x} = (x, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1,\ \alpha \in \mathbb{R}^1. \tag{4.7}$$

Fixed, "vertical" cross-section for (4.7) for *all* α (near 0)

$$\Pi = L \times \mathbb{S}^1 = \{(\tilde{x} = (x, \theta) \in X : \ x \in L,\ \theta \in \mathbb{S}^1\}$$

where

$$L: \;\; x = x_0^0 + \beta \, \frac{\eta_0}{\|\eta_0\|}, \quad |\beta| < \varepsilon,$$

$$\beta = \frac{\langle \, \eta_0 \, , x - x_0^0 \, \rangle}{\|\eta_0\|}, \quad \eta_0 = f_0^{\perp}(x_0^0) \in \mathbb{R}^2, \quad x_0^0 = x^0(0).$$

Transversal intersections of the stable and unstable manifolds $\tilde{W}_\alpha^s$ and $\tilde{W}_\alpha^u$ with the “vertical” cross-section Π

$$(s(\theta, \alpha), \, \theta \; (\mathrm{mod} \; 2\pi)) \in \Pi, \quad \theta \in \mathbb{S}^1,$$

$$(u(\theta, \alpha), \, \theta \; (\mathrm{mod} \; 2\pi)) \in \Pi, \quad \theta \in \mathbb{S}^1,$$

The **split function** (“M version”)

$$\beta(\theta, \alpha) = \frac{\langle \, \eta_0 \, , u(\theta, \alpha) - s(\theta, \alpha) \, \rangle}{\|\eta_0\|}$$

measures the *signed* “horizontal” distance (i.e. at $\theta =$ constant) *from* $\tilde{W}_\alpha^s$ *to* $\tilde{W}_\alpha^u$, along Π.

The **Melnikov function**, is the numerator of $\beta(\theta, \alpha)$,

$$M(\theta, \alpha) = \langle \, \eta_0 \, , \, u(\theta, \alpha) - s(\theta, \alpha) \, \rangle$$

For $\alpha = 0$, $\tilde{W}_0^s$ and $\tilde{W}_0^u$ coincide along Π,

$$M(\theta, 0) = 0 \quad \text{for all } \theta \in \mathbb{S}^1$$

and therefore we can “factor out” α from the Taylor expansion of $M(\theta, \alpha)$

about $\alpha = 0$

$$M(\theta,\alpha) = \alpha\,\tilde{M}(\theta,\alpha), \qquad \tilde{M}(\theta,\alpha) = M_\alpha(\theta,0) + O(|\alpha|)$$

Now we define two "time-dependent Melnikov functions"

$$M^u(t,\theta,\alpha) = \langle\, f_0^\perp(x^0(t)),\, x^u(t,\theta,\alpha)\,\rangle, \quad -\infty < t \le 0,$$

$$M^s(t,\theta,\alpha) = \langle\, f_0^\perp(x^0(t)),\, x^s(t,\theta,\alpha)\,\rangle, \quad 0 \le t < +\infty.$$

When $t = 0$, we have

$$M(\theta,\alpha) = M^u(0,\theta,\alpha) - M^s(0,\theta,\alpha).$$

Taking the α-derivative at $\alpha = 0$, and the t-derivative at any t (using result of HW 2 problem 1, etc.) after a long calculation we have

$$\dot{M}^u_\alpha(t,\theta,0) = \langle\, f_0^\perp(x^0(t)),\, f_1(x^0(t),\omega t+\theta)\,\rangle$$

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Now we integrate, to recover $M^u_\alpha(0,\theta,0)$

By the fundamental theorem of calculus,

so we have

We calculate

and we have

$$M^u_\alpha(0, \theta, 0) = \int_{-\infty}^{0} \langle\, f_0^\perp(x^0(t)),\ f_1(x^0(t), \omega t + \theta)\,\rangle\, dt.$$

Similarly (**Exercise**),

$$M^s_\alpha(0, \theta, 0) = -\int_{0}^{\infty} \langle\, f_0^\perp(x^0(t)),\ f_1(x^0(t), \omega t + \theta)\,\rangle\, dt$$

and finally

which is the Melnikov integral (4.8).

Summary/review

Theorem 4.2. ($\dot{\theta} = \omega > 0,\ \ \theta \in \mathbb{S}^1$)

$$M_\alpha(\theta, 0) = \int_{-\infty}^{+\infty} \langle\, f_0^\perp(x^0(t)), f_1(x^0(t), \omega t + \theta) \,\rangle \, dt$$

If (M.0) and (M.1):

Theorem 4.3. ($\dot{\gamma} = 0,\ \ \gamma \in \mathbb{R}^1$)

$$M_\alpha(\gamma, 0) = \int_{-\infty}^{+\infty} \langle\, f_0^\perp(x^0(t)), f_1(x^0(t), \gamma) \,\rangle \, dt$$

If (M.0) and (M.2):

Generalizations of Theorems 4.2 and 4.3.

Melnikov's method can be generalized to work in situations where the unperturbed system is not Hamiltonian. For example, a useful generalization can be made in the setting of Theorem 4.1 (the Andronov-Leontovich theorem on the homoclinic bifurcation in $\mathbb{R}^2$): recall we have a 1-parameter family of 2-dimensional vector fields (autonomous ODEs)

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^2, \ \alpha \in \mathbb{R}^1. \tag{4.1}$$

and the unperturbed ($\alpha = \alpha_0$) vector field satisfies

$$f(p_0^0, \alpha_0) = 0 \qquad\qquad (\textit{equilibrium}), \tag{SL.0.i}$$

$$A_0 = f_x(p_0^0, \alpha_0) \text{ has eigenvalues } \lambda_{10} < 0 < \lambda_{20} \qquad (\textit{hyperbolic saddle}), \tag{SL.0.ii}$$

$$\dot{x} = f(x, \alpha_0) \text{ has an orbit } \Gamma = \{x^0(t)\} \text{ that is homoclinic to } p_0^0 \qquad (\textit{bifurcation}). \tag{SL.0.iii}$$

In Theorem 4.1, if the unperturbed vector field $f(\,\cdot\,, \alpha_0)$ is not Hamiltonian (if it is Hamiltonian, then (SL.2) *cannot* be satisfied), consider a Melnikov integral

$$M_\alpha(\alpha_0) = \int_{-\infty}^{+\infty} \langle\, \eta(t), f_\alpha(x^0(t), \alpha_0)\,\rangle \, dt,$$

where (see the textbook, p. 211)

$$\eta(t) = \exp\left[-\int_0^t \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)\bigg|_{(x,\alpha)=(x^0(\tau),\alpha_0)} d\tau\right] f^{\perp}(x^0(t), \alpha_0),$$

and the condition on the derivative of the ("A-L") split function (SL.1) is equivalent to an analytic expression (for the "M" split function)

$$M_\alpha(\alpha_0) \neq 0. \tag{SL.1'}$$

Also, the vector function $\eta(t)$ can be characterized as the unique-up-to-a-scalar-multiple bounded solution of the "adjoint variational equation"

$$\dot{\eta} = -A(t)^\intercal \eta, \qquad \eta \in \mathbb{R}^2, \quad t \in \mathbb{R}$$

(i.e. the adjoint variational equation has a 1-dimensional function space of bounded solutions, span$\{\eta(t)\}$), where $A(t) = f_x(x^0(t), \alpha_0)$ is the linearization of the unperturbed vector field at the homoclinic solution.

For perturbations of systems with homoclinic orbits in dimensions $n \geq 3$, and also for perturbations of systems with heteroclinic solutions, generalized Melnikov integrals are defined in terms of an inner product with an $\eta(t) \in \mathbb{R}^n$, a unique-up-to-a-scalar-multiple bounded solution of the adjoint variational equation. This approach generalizes to infinite dimensions (PDEs, DDEs), and is useful, e.g. for studying stability and bifurcation of travelling waves in reaction-diffusion systems.

Transverse homoclinic points and homoclinic tangles

We return to the setting of Theorem 4.2. Suppose, for the 3-dimensional autonomous system (4.7) that (M.0) and (M.1) hold, so for any sufficiently small $\alpha \neq 0$, Theorem 4.2 gives the existence of some $\hat{\theta}(\alpha) \in \mathbb{S}^1$ such that the split function $\beta(\hat{\theta}(\alpha), \alpha) = 0$.

Therefore the Poincaré map $\hat{P}(\,\cdot\,, \alpha)$ based on the cross-section

has a **transverse homoclinic point** $\hat{q}_0$, a point where the stable and unstable manifolds for $\hat{P}(\,\cdot\,, \alpha)$, of its the hyperbolic saddle fixed point $\hat{p}^0(\alpha)$, have a transversal intersection (the tangent vectors to the manifolds at $\hat{q}_0$ are linearly independent).

Since $\hat{q}_0 \in W^s(\hat{p}^0(\alpha))$, and since $W^s(\hat{p}^0(\alpha))$ is an invariant manifold, all the iterates under $\hat{P}(\,\cdot\,,\alpha)$ satisfy

$$\hat{q}_k \in W^s(\hat{p}^0(\alpha)), \quad \text{for all } k \in \mathbb{Z}.$$

Similarly,

$$\hat{q}_k \in W^u(\hat{p}^0(\alpha)), \quad \text{for all } k \in \mathbb{Z},$$

thus

$$\hat{q}_k \in W^s(\hat{p}^0(\alpha)) \cap W^u(\hat{p}^0(\alpha)), \quad \text{for all } k \in \mathbb{Z}.$$

Furthermore, the fact that $\hat{P}(\,\cdot\,,\alpha)$ is a local diffeomorphism implies

Near the hyperbolic saddle fixed point $\hat{p}^0(\alpha)$, the Poincaré map $\hat{P}(\,\cdot\,,\alpha)$ expands distances in the unstable direction and contracts distances in the stable direction, and one can prove that this results in a "homoclinic tangle" (see, e.g. Wiggins)

Ingredients of chaos

Consider a dynamical system, with either continuous time ($\dot{x} = f(x)$, $x(t) = \varphi^t(x_0)$) or discrete time ($x \mapsto f(x)$, $x_k = f^k(x_0)$), and suppose Λ is an invariant set.

a. Sensitive dependence

The dynamical system has **sensitive dependence** on Λ, if there is some $\varepsilon_0 > 0$ such that, for any $x \in \Lambda$ and any $\delta > 0$, there is always some $y \in \Lambda$ with $\|x - y\| < \delta$ and

$$\|\varphi^t(x) - \varphi^t(y)\| \geq \varepsilon_0 \quad \text{for some } t > 0,\, t \in \mathbb{R}$$

or

$$\|f^k(x) - f^k(y)\| \geq \varepsilon_0 \quad \text{for some } k > 0,\, k \in \mathbb{Z}$$

b. Topological transitivity

A subset U of a closed set Λ in $\mathbb{R}^n$ is **relatively open** in Λ, if it is the intersection of Λ with an open subset of $\mathbb{R}^n$. The dynamical system is **topologically transitive** on a closed invariant set Λ, if for any two relatively open subsets U, V in Λ we have

$$\varphi^t(U) \cap V \neq \emptyset \quad \text{for some } t > 0,\, t \in \mathbb{R}$$

or

$$f^k(U) \cap V \neq \emptyset \quad \text{for some } k > 0,\, k \in \mathbb{Z}$$

e.g. for any point in Λ, its forward orbit eventually visits arbitrarily close to every other point in Λ, or "wanders everywhere" in Λ.

Some examples of topologically transitive invariant sets

For flows:

i. $\Lambda = \{$ an equilibrium in $\mathbb{R}^n$ $\}$ is a topologically transitive invariant set.

ii. $\Lambda = \{$ a cycle in $\mathbb{R}^n$ $\}$ is a topologically transitive invariant set.

iii.
$$\dot{\theta}_1 = 1,\ \ \dot{\theta}_2 = \omega, \quad (\theta_1, \theta_2) \in \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2$$

if ω is irrational, then all of the 2-torus $\Lambda = \mathbb{T}^2$ is a topologically transitive invariant set.

For maps:

i. $\Lambda = \{$ a fixed point in $\mathbb{R}^n$ $\}$ is a topologically transitive invariant set.

ii. $\Lambda = \{$ a cycle in $\mathbb{R}^n$ $\}$ is a topologically transitive invariant set.

iii. See HW 1, problem 4, $x \mapsto Ax$, $x \in \mathbb{R}^2$ discrete rotation:

$$A = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}.$$

If $\phi/2\pi$ is irrational ($\theta \mapsto \theta + \phi$ in $\mathbb{S}^1$), then the circle $\Lambda = \{\, x \in \mathbb{R}^2 : \|x\| = 1 \,\} \cong \mathbb{S}^1$ is a topologically transitive invariant set.