MATH 552 (2023W1) Lecture 34: Fri Dec 1

[**Last lecture:** ... Melnikov's method (derivation). Generalizations. Transverse homoclinic points and homoclinic tangles ...]

We returned to the setting of Theorem 4.2. Suppose, for the 3-dimensional autonomous system (4.7) that (M.0) and (M.1) hold, so for any sufficiently small $\alpha \neq 0$, Theorem 4.2 gives the existence of some

$$\hat{\theta}(\alpha) = \theta_0 + O(|\alpha|) \in \mathbb{S}^1$$

such that the split function $\beta(\hat{\theta}(\alpha), \alpha) = 0$ (exactly, for all α near 0).

Therefore the Poincaré map $\hat{P}(\,\cdot\,,\alpha)$ based on the cross-section

$$\Sigma_{\hat{\theta}(\alpha)} = \{\, (x,\theta) \in X : \ x \in \mathbb{R}^2, \ \theta = \hat{\theta}(\alpha) \ (\mathrm{mod}\ 2\pi) \,\}$$

has a **transverse homoclinic point** $\hat{q}_0$, a point where the stable and unstable manifolds for $\hat{P}(\,\cdot\,,\alpha)$, of its the hyperbolic saddle fixed point $\hat{p}^0(\alpha)$, have a transversal intersection (the tangent vectors to the manifolds at $\hat{q}_0$ are linearly independent).

It follows that the entire orbit of $\hat{q}_0$ under the Poincaré map $\hat{P}(\,\cdot\,,\alpha)$ consists of intersections of the stable and unstable manifolds

$$\hat{q}_k \in W^s(\hat{p}^0(\alpha)) \cap W^u(\hat{p}^0(\alpha)), \quad \text{for all } k \in \mathbb{Z}.$$

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Furthermore, the fact that $\hat{P}(\,\cdot\,,\alpha)$ is a local diffeomorphism implies all the inersections at $\hat{q}_k$ are transversal.

Near the hyperbolic saddle fixed point $\hat{p}^0(\alpha)$, the Poincaré map $\hat{P}(\,\cdot\,,\alpha)$ expands distances in the unstable direction and contracts distances in the stable direction, and it can be proved that this results in a "homoclinic tangle" (see, e.g. Wiggins)

Ingredients of chaos

Consider a dynamical system, with either continuous time ($\dot{x} = f(x)$, $x(t) = \varphi^t(x_0)$) or discrete time ($x \mapsto f(x)$, $x_k = f^k(x_0)$), and suppose Λ is an invariant set for the dynamical system.

a. Sensitive dependence

The dynamical system has **sensitive dependence** (on initial conditions) on Λ, if there is some $\varepsilon_0 > 0$ such that, for any $x \in \Lambda$ and any $\delta > 0$, there is always some $y \in \Lambda$ with $\|x - y\| < \delta$ and

$$\|\varphi^t(x) - \varphi^t(y)\| \geq \varepsilon_0 \quad \text{for some } t > 0,\, t \in \mathbb{R}$$

or

$$\|f^k(x) - f^k(y)\| \geq \varepsilon_0 \quad \text{for some } k > 0,\, k \in \mathbb{Z}$$

b. Topological transitivity

A subset U of a closed set Λ in $\mathbb{R}^n$ is **relatively open** in Λ, if it is the intersection of Λ with an open subset of $\mathbb{R}^n$. The dynamical system is **topologically transitive** on a closed invariant set Λ, if for any two relatively open subsets U, V in Λ we have

$$\varphi^t(U) \cap V \neq \emptyset \quad \text{for some } t > 0,\, t \in \mathbb{R}$$

or

$$f^k(U) \cap V \neq \emptyset \quad \text{for some } k > 0,\, k \in \mathbb{Z}$$

e.g. for any point in Λ, its forward orbit eventually visits arbitrarily close to every other point in Λ, or "wanders everywhere" in Λ.

Some examples of topologically transitivity on invariant sets:

For flows

i. A flow in $\mathbb{R}^n$ is topologically transitive on $\Lambda = \{$ an equilibrium $\}$.

ii. A flow in $\mathbb{R}^n$ is topologically transitive on $\Lambda = \{$ a cycle $\}$.

iii.

$$\dot{\theta}_1 = 1, \;\; \dot{\theta}_2 = \omega, \quad (\theta_1, \theta_2) \in \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2$$

If ω is irrational, then the flow is topologically transitive on all of the 2-torus $\Lambda = \mathbb{T}^2$.

For maps

i. A map in $\mathbb{R}^n$ is topologically transitive on $\Lambda = \{$ a fixed point $\}$.

ii. A map in $\mathbb{R}^n$ is topologically transitive on $\Lambda = \{$ a cycle $\}$.

iii. See HW 1, problem 4, $x \mapsto Ax$, $x \in \mathbb{R}^2$ a discrete rotation where

$$A = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}.$$

If $\phi/2\pi$ is irrational ($\theta \mapsto \theta + \phi$ in $\mathbb{S}^1$), then map restricted to the unit circle is topologically transitive on $\Lambda = \{\, x \in \mathbb{R}^2 : \ \|x\| = 1 \,\} \cong \mathbb{S}^1$.

c. Compactness

A subset of $\mathbb{R}^n$ (or of any finite-dimensional manifold) is **compact**, if it is closed and bounded. It can be proved that a dynamical system on a compact invariant set Λ is topologically transitive if and only if there is a forward orbit that is **dense** in Λ:

i.e. there is some x_0 in Λ such that, for any y_0 in Λ and $\varepsilon > 0$, there exists some $t > 0$ (or some $k > 0$) such that

$$\|\varphi^t(x_0) - y_0\| < \varepsilon \quad (\text{or } \ \|f^k(x_0) - y_0\| < \varepsilon\,).$$

E.g. there is a forward orbit in Λ that eventually visits arbitrarily close to every other point in Λ, or "wanders everywhere" in Λ.

Chaos

A dynamical system restricted to an invariant set Λ (or the invariant set Λ itself) is **chaotic** if

a. the dynamical system has sensitive dependence on Λ,

b. the dynamical system is topologically transitive on Λ,

c. Λ is compact.

Theorem 4.4. (**Smale & Birkhoff**) *If a smooth map $x \mapsto f(x)$, $x \in \mathbb{R}^n$, $n \geq 2$, has a hyperbolic fixed point p^0 and there exists a point $q_0 \neq p^0$ of transversal intersection between the stable manifold $W^s(p^0)$ and unstable manifold $W^u(p^0)$, then there exists a positive integer N such that f^N has a compact invariant set Λ, containing a countably infinite number of cycles of arbitrarily long period and an uncountably infinite number of nonperiodic orbits, on which f^N is chaotic.*

Thus, Melnikov's method (Theorem 4.2) can be used to show that a specific system (such as the nonlinear oscillator Example 4.A) has a Poincaré map with a chaotic invariant set (i.e. the system "has chaos").

Another example: the vector field in HW 5 problems 2 and 3 arises as an *approximation*, in the analysis of a vector field in $\mathbb{R}^n$, $n \geq 3$, near a limit cycle that loses stability, as two parameters are varied, due to a generic double-one eigenvalue for the Poincaré map linearized at the fixed point corresponding to the limit cycle. So a homoclinic orbit in the 2-dimensional vector field may imply chaos in the higher-dimensional vector field (but the analysis is quite involved and the results are subtle).

Ideas from the proof of the Smale-Birkhoff theorem; the Smale horseshoe

The Smale horseshoe map is discussed in the textbook, section 1.3.2.

The Smale-Birkhoff theorem (Theorem 4.4) has been described as

$$\text{“fish”} \quad \Rightarrow \quad \text{“horseshoe”}$$