[**Last lecture:** ... transverse homoclinic points and homoclinic tangles. Ingredients of chaos (sensitive dependence, topological transitivity, compactness). Chaos ...]

Theorem 4.4. (**Smale & Birkhoff**) *If a smooth map* $x \mapsto f(x)$, $x \in \mathbb{R}^n$, $n \geq 2$, *has a hyperbolic fixed point* p^0 *and there exists a point* $q_0 \neq p^0$ *of transversal intersection between the stable manifold* $W^s(p^0)$ *and unstable manifold* $W^u(p^0)$, *then there exists a positive integer* N *such that* f^N *has a compact invariant set* Λ, *containing a countably infinite number of cycles of arbitrarily long period and an uncountably infinite number of nonperiodic orbits, on which* f^N *is chaotic.*

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Ideas from the proof of the Smale-Birkhoff theorem; the Smale horseshoe

The Smale horseshoe map is discussed in the textbook, section 1.3.2.

The Smale-Birkhoff theorem (Theorem 4.4) has been described as

$$\text{“fish”} \quad \Rightarrow \quad \text{“horseshoe”}$$

Some other generic global bifurcations with one parameter

Saddle connection (heteroclinic) bifurcation for a flow in $\mathbb{R}^2$

Bifurcation condition: if $\alpha = \alpha_0$, there is a heteroclinic orbit connecting a pair of hyperbolic saddles.

Example 4.B. (Textbook, p. 58.) In $\mathbb{R}^2$:

$$\dot{x}_1 = 1 - x_1^2 - \alpha\, x_1 x_2,$$

$$\dot{x}_2 = x_1 x_2 + \alpha\,(1 - x_1^2).$$

Saddle-node (fold) bifurcation of a limit cycle for a flow in $\mathbb{R}^n$, $n \geq 2$

Bifurcation condition: if $\alpha = \alpha_0$, there is a nonhyperbolic limit cycle with simple *nontrivial* Floquet multiplier 1.

Example 4.C. In $\mathbb{R}^2$:

$$\dot{x}_1 = -x_1 - x_2 + \alpha\, x_1(x_1^2 + x_2^2) - x_1(x_1^2 + x_2^2)^2,$$
$$\dot{x}_2 = x_1 - x_2 + \alpha\, x_2(x_1^2 + x_2^2) - x_2(x_1^2 + x_2^2)^2.$$

Or, in polar coordinates

$$\dot{r} = r\,(-1 + \alpha\, r^2 - r^4), \quad \dot{\theta} = 1.$$

Period-doubling bifurcation of a limit cycle for a flow in $\mathbb{R}^n$, $n \geq 3$

Bifurcation condition: if $\alpha = \alpha_0$, there is a nonhyperbolic limit cycle with simple (nontrivial) Floquet multiplier -1.

E.g.

SNIC bifurcation for a flow in $\mathbb{R}^2$

Bifurcation condition: if $\alpha = \alpha_0$, there is a saddle-node equilibrium in an invariant circle. (Textbook: saddle-node homoclinic, p. 250.)

Example 4.D. In $\mathbb{R}^2$:

$$\dot{x}_1 = x_1(1 - x_1^2 - x_2^2) - x_2(\alpha - x_1),$$

$$\dot{x}_2 = x_1(\alpha - x_1) + x_2(1 - x_1^2 - x_2^2).$$

Or, in polar coordinates

$$\dot{r} = r\,(1 - r^2), \quad \dot{\theta} = \alpha - r\cos(\theta).$$

Invariant torus (Neimark-Sacker) bifurcation of a limit cycle for a flow in $\mathbb{R}^n$, $n \geq 3$

Bifurcation condition: if $\alpha = \alpha_0$, there is a nonhyperbolic limit cycle with nontrivial Floquet multipliers $e^{\pm i\phi}$, $e^{ik\phi} \neq 1$ for $k = 1, 2, 3, 4$. (Textbook, p. 131; p. 136.)

E.g.

Period doubling cascade (to chaos) for a map in $\mathbb{R}^1$

Example 4.E. (See HW 4 problem 1. Also, textbook p. 129; p. 145; Wikipedia article "logistic map"; Strogatz)

In $\mathbb{R}^1$:

$$x \mapsto \alpha x(1-x)$$