Sturm-Liouville theory

ODE: $\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y + \lambda\sigma(x)y = 0$

Solved on $[a,b]$ with $p,\sigma \geq 0$ on $[a,b]$.

Possible BCs: (i) $c_1 y(a) + c_2 y'(a) = 0$

$d_1 y(b) + d_2 y'(b) = 0$

for constants c_1, c_2, d_1, d_2.

(ii) Regularity (i.e. y must be regular) if $p(a)=0$ or $p(b)=0$

(iii) Periodic: $y(a)=y(b)$, $y'(a)=y'(b)$.

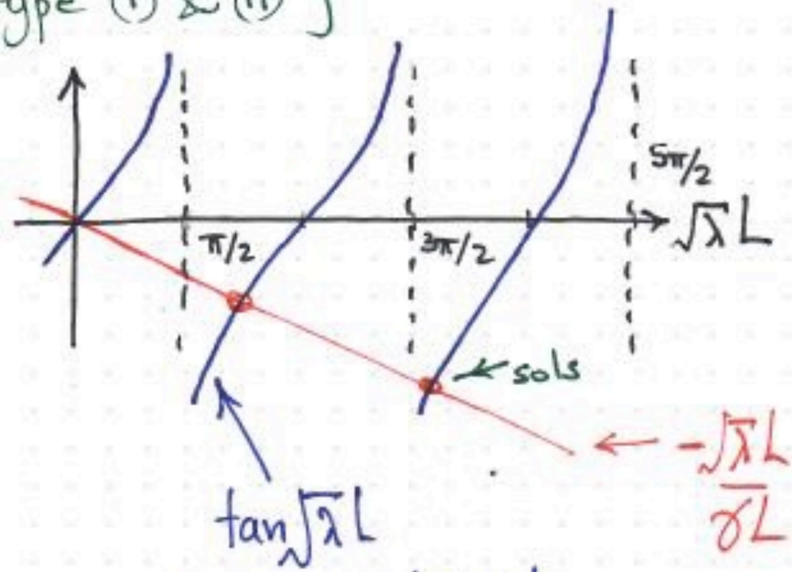
[Can mix & match conditions of type (i) & (ii)]

e.g. $u_t = u_{xx}$

$u(0,t)=0$

$-\gamma u(x,t) = u_x(x,t)$ at $x=L$

$u(x,0)=f(x)$



Sol $\longrightarrow$ $u = \sum_{n=1}^{\infty} b_n \sin k_n x \, e^{-k_n^2 t}$

with $\lambda = k_n^2$ the solutions to $\tan k_n L = -k_n/\gamma$

* $\lambda = 0$ is trivial
* $k_n \approx (n-\frac{1}{2})\pi/L$ for $n \gg 1$

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* the solution pairs $\{\lambda_n, y_n(x)\}_{n=1,2,\ldots}$ form an infinite sequence
* $\lambda_1 < \lambda_2 < \lambda_3 \ldots$ with $\lambda_n \to \infty$ for $n \to \infty$. (as seen above)
* y_n has exactly n-1 zeros between $x=a$ and $x=b$.
* the set $\{y_n(x)\}$ is complete. i.e. for any (Continuous) func

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$$

Justifies how we can represent the PDE in terms of the y_n's!