* the S.L. "eigenfunctions", $y_n(x)$, are <u>orthogonal</u> :

$$\int_a^b y_n(x)\, y_m(x)\, \sigma(x)\, dx = 0 \quad \text{f } n \neq m.$$

$\{ \mathcal{L} y_n - \lambda_n y_n = 0 \leftarrow$ SL ODE with $\mathcal{L} \equiv -\frac{d}{dx} p \frac{d}{dx} - q$

Also $\mathcal{L} y_m - \lambda_m y_m = 0$. (cf. matrix eigenvalue problem $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$)

So $\int_a^b y_m \mathcal{L} y_n\, dx - \lambda_n \int y_n y_m \sigma\, dx = 0$.

But $\int_a^b y_m \frac{d}{dx}\left(p \frac{dy_n}{dx}\right) dx = \int_a^b y_n \frac{d}{dx}\left(p \frac{dy_m}{dx}\right) dx$

after two integration by parts, and for all 3 types of BCs.

Hence $\int_a^b y_m \mathcal{L} y_n\, dx = \int_a^b y_n \mathcal{L} y_m\, dx = \lambda_m \int y_n y_m \sigma\, dx$

Altogether... $(\lambda_n - \lambda_m) \int_a^b y_n y_m \sigma\, dx = 0$

establishing the result, as long as $n \neq m$. $\}$

<u>Finally</u> $\quad f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$

multiply by $y_m \sigma$ & integrate

$$\rightarrow \int_a^b f y_m \sigma\, dx = \sum_{n=1}^{\infty} c_n \int_a^b y_n y_m \sigma\, dx$$

$$= c_m \int_a^b y_m^2 \sigma\, dx$$

So,

$$\boxed{c_n = \frac{\int_a^b f(x)\, y_n(x)\, \sigma(x)\, dx}{\int_a^b [y_n(x)]^2 \sigma(x)\, dx}}$$

So we even have a formula for the coeffs of the series ....