$\boxed{\text{Oscillations of a Drum}}$

Problem: $u_{tt} = \nabla^2 u$ for $u(r,\theta,t)$
$(c=1)$

$$\left\{\begin{array}{l} u \text{ } 2\pi\text{-periodic in } \theta \\ u \text{ regular for } r \to 0 \\ u = 0 \text{ at } r = 1 \\ \left.\begin{array}{l} u = f(r,\theta) \\ u_t = g(r,\theta) \end{array}\right\} \text{at } t = 0. \end{array}\right.$$

Sep. of Var: $u = R(r)\Theta(\theta)T(t)$

$$\rightsquigarrow \underbrace{\frac{1}{rR}(rR')' + \frac{1}{r^2}\frac{\Theta''}{\Theta}}_{\text{fun.}(r,\theta)} = \underbrace{\frac{T''}{T}}_{\text{fun.}(t)} = -\lambda$$

(first sep. const.)
gives $\cos\omega t, \sin\omega t$ if $\lambda = \omega^2$
$a + bt$ if $\lambda = 0$.

Then

$$\underbrace{\frac{r}{R}(rR')' + \lambda r^2}_{\text{fun}(r)} = \underbrace{-\frac{\Theta''}{\Theta}}_{\text{fun}(\theta)} = m^2$$

(second sep. const.)

gives Θ as $\cos m\theta, \sin m\theta$ if $m = 1, 2, \ldots$
or const if $m = 0$ ← 2π-periodic in θ!
i.e. a Fourier series in angle

Last,

$$R'' + \frac{1}{r}R' + \lambda R - \frac{m^2}{r^2}R = 0$$

or

$$(rR')' + \lambda r R - \frac{m^2}{r}R = 0$$

this is a SL ODE with
$(x,y) \to (r,R)$
$p(r) = s(r) = r$
(non-negative)
$q = -m^2/r$

BC's: $R(1) = 0$ ← type (i) condition
R regular for $r \to 0$ ← type (ii) condition.
$(p \to 0)$

i.e. we have a SL problem with sols. λ_{mn} or λ^m_n
$R_{mn}(r)$ or $R^m_n(r)$

for those who like subscripts

for those who like superscripts.

Since m appears as a <u>parameter</u>.
i.e. we have a SLP for each m

Structure of solution:

$$u(r,\theta,t) = \underbrace{\sum}_{\text{all possible } n\text{'s}} \underbrace{\sum}_{\text{all possible } m\text{'s}} R^m_n(r) \left\{\begin{array}{l} \cos m\theta \\ \sin m\theta \\ \text{const}, m=0 \end{array}\right\} \left\{\begin{array}{l} \cos\omega t \\ \sin\omega t \\ a+bt, \omega=0 \end{array}\right\} \left(\because\right)^m_n$$

$\omega^2 = \lambda^m_n$

choose these using the ICs, assisted by SL expansion formulae → arb. const. for each sol.