$u_{tt} = \nabla^2 u$, $u = 0$ at $r = 1$,
$u$ reg. for $r \to 0$, $2\pi$-periodic in $\theta$

Big, bad book-keeping exercise...

$\left.\begin{array}{l} u = f(r,\theta) \\ u_t = g(r,\theta) \end{array}\right\}$ at $t = 0$.

Gen Sol. by Sep. of Var....

$$u = \sum_{n=1}^{\infty} \left(\tfrac{1}{2} a_{0n} \cos z_n^0 t + \tfrac{1}{2} \tilde{a}_{0n} \sin z_n^0 t\right) J_0(z_n^0 r)$$

Both a Fourier series and a SL eigenfunk exp.

$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \Big[ \left(a_{mn} \cos z_n^m t + \tilde{a}_{mn} \sin z_n^m t\right) \cos m\theta \, J_m(z_n^m r) + \left(b_{mn} \cos z_n^m t + \tilde{b}_{mn} \sin z_n^m t\right) \sin m\theta \, J_m(z_n^m r) \Big]$$

eg. $f = f(r)$, $g = 0$ $\qquad (\tilde{a}_{0n} = a_{mn} = \tilde{a}_{mn} = b_{mn} = \tilde{b}_{mn} = 0)$

$$\Rightarrow u = \sum_{n=1}^{\infty} \underbrace{A_n}_{\frac{1}{2} a_{0n}} \cos z_n^0 t \, J_0(z_n^0 r)$$

& by SL expansion theorem, $\quad A_n = \dfrac{\int_0^1 f(r) \overbrace{J_0(z_n^0 r)}^{\text{SL eigenfunk}} \overbrace{r}^{\sigma(r) = r} dr}{\int_0^1 [J_0(z_n^0 r)]^2 r \, dr}$

eg. $f = 0$, $g = G(r) \cos M\theta$

$a_{0n} = a_{mn} = b_{mn} = \tilde{a}_{0n} = \tilde{b}_{mn} = 0$
and $\tilde{a}_{mn} = 0$ unless $m = M$.

$$\Rightarrow u = \sum_{n=1}^{\infty} B_n \sin z_n^M t \, J_M(z_n^M r) * \cos M\theta$$

$$z_n^M B_n = \frac{\int_0^1 G(r) J_M(z_n^M r) r \, dr}{\int_0^1 [J_M(z_n^M r)]^2 r \, dr}$$

from t-deriv of $\sin z_n^M t$ $\qquad (B_n \equiv \tilde{a}_{Mn})$

Last, take $g = 0$, $f = f(r,\theta)$

Put $f(r,\theta) = \frac{1}{2}\alpha_0(r) + \sum_{m=1}^{\infty} \left[\alpha_m(r) \cos m\theta + \beta_m(r) \sin m\theta\right]$
(expanding $f$ as a F.S.)

Require

$$\alpha_0 = \sum_{n=1}^{\infty} a_{0n} J_0(z_n^0 r) \longrightarrow a_{0n} = \frac{\int_0^1 \alpha_0 J_0(z_n^0 r) r \, dr}{\int_0^1 [J_0(z_n^0 r)]^2 r \, dr}$$

$$\alpha_m = \sum_{n=1}^{\infty} a_{mn} J_m(z_n^m r)$$

$$\beta_m = \sum_{n=1}^{\infty} b_{mn} J_m(z_n^m r)$$

$$\left\{\begin{array}{c} a_{mn} \\ b_{mn} \end{array}\right\} = \frac{\int_0^1 \left\{\begin{array}{c} \alpha_m \\ \beta_m \end{array}\right\} J_m(z_n^m r) r \, dr}{\int_0^1 [J_m(z_n^m r)]^2 r \, dr}$$