Another Example ...

$\nabla^2 u = 0$, $u = u(\rho, \theta, \varphi)$, u regular (at $\rho = 0$, $\theta = 0$ & π)

$u = f(\theta, \varphi)$ at $\rho = 1$, 2π-periodic in φ.

(spherical polar coords)

$$\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2} = 0$$

Sep. Var: $u = R(\rho)\Theta(\theta)\Phi(\varphi) \Rightarrow$

$$\frac{\Phi''}{\Phi} = \underset{\text{(Four. Ser. in } \varphi)}{\text{const.} \longrightarrow -m^2} \quad \text{so that} \quad \Phi = \begin{cases} \text{const.} \times \begin{cases} \cos m\varphi \\ \sin m\varphi \end{cases} & m = 1, 2 \ldots \\ \text{const}, & m = 0 \end{cases}$$

Then

$$\cancel{\frac{1}{\rho^2}}\frac{(\rho^2 R')'}{R} + \cancel{\frac{1}{\rho^2}}\left[\frac{1}{\sin\theta}\frac{(\sin\theta\,\Theta')'}{\Theta} - \frac{m^2}{\sin^2\theta}\right] = 0$$

— must both equal another separation const.

Put

$$(\sin\theta\,\Theta')' - \frac{m^2\,\Theta}{\sin\theta} + \lambda \sin\theta\,\Theta = 0$$

This is a SLP with type (ii) (regularity) BCs.

$$(x, y, p, s, q) \longleftrightarrow (\theta, \Theta, \sin\theta, \sin\theta, -m^2/\sin\theta)$$

Establishes that the sols are $\{\lambda_n, \Theta_n\}$ with $\lambda_n > 0$

($\sin\theta$ is non-negative for $0 \leq \theta \leq \pi$)

Finally, $\rho^2 R'' + 2\rho R' - \lambda R = 0$

This is an Euler eq. with sols, $R = C\rho^\alpha$

$$\longrightarrow \quad \alpha^2 + \alpha - \lambda = 0.$$

could solve for α in terms of λ, but this looks nicer

So if $\lambda \equiv \nu(\nu+1)$, we have $\alpha = \nu$ or $-\nu-1$

$$\Rightarrow u = \sum_{\text{all } n}{}' \sum_{\text{all } m}{}' \left\{\rho^\nu \text{ or } \rho^{-\nu-1}\right\}\left\{\Theta_n(\theta)\right\}\begin{Bmatrix} \sin m\varphi \\ \cos m\varphi \\ \text{const.} \end{Bmatrix} \times \text{arb. const.}$$