$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \underbrace{\nu(\nu+1)}_{\lambda} y - \frac{m^2}{1-x^2} y = 0 \qquad (*)$$

ODE for $\Theta(\theta)$ after the change of variable

$$x = \cos\theta$$

$$-1 < x < 1, \qquad \begin{array}{l} x = -1 \leftrightarrow \theta = \pi \\ x = +1 \leftrightarrow \theta = 0 . \end{array}$$

First look at $m=0$

$$\left.\begin{array}{lll} y = 1, & \nu = 0 & (\lambda = 0) \\ y = x, & \nu = 1 & (\lambda = 2) \\ y = \frac{3}{2}x^2 - \frac{1}{2}, & \nu = 2 & (\lambda = 6) \end{array}\right\}$$ possible sols that one can almost guess.

e.g. $\nu = 0$ gives $\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] = 0 \rightarrow y = C_1 + C_2 \ln\frac{1+x}{1-x}$

Regular Irregular.

Could it be that $\nu = n = 0, 1, 2, \ldots$

& $y =$ polyn. of degree n ?

$P_n(x)$

YES! These are Legendre polynomials & $(*)$ is Legendre's ODE.

N.B. The quoted solutions use the standard normalization

$P_n(1) = 1$ (choice i.e.)

Actual solution is $P_n(x)$ times an arbitrary const.

More generally, look for a series sol:

$$y = P_n(x) = \sum_{m=0}^{\infty} a_m x^m$$ ⟵ imperative not to use n as the summing variable!

Plugging in...

$$\sum_{m=0}^{\infty}\left[m(m-1)a_m x^{m-2} - m(m+1)a_m x^m + n(n+1)a_m x^m\right] = 0$$

put $m-2 = ☺$ here, then replace ☺ by m

$$\boxed{a_{m+2} = \frac{-n(n+1) + m(m+1)}{(m+1)(m+2)}\, a_m}$$